Malcolm S. Barton
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Hand in
Fords College Algebra
Analytic Geometry - Siceloff
Wentworth

Value of,

(1) $\tan 128^\circ 42' + \cot 114^\circ 37' - \sin 127^\circ 22'$ $+ \tan 112^\circ 13' - \cos 98^\circ 28'$

(2) $\sec(-120^\circ) + \csc(-240^\circ)$ $+ \tan 480^\circ - \cos 1200^\circ$

(1) Prove $f(90^\circ + \theta) = \pm \operatorname{cof} \theta$

(2) " $f(180^\circ - \theta) = \pm f(\theta)$

PLANE TRIGONOMETRY AND NUMERICAL COMPUTATION

(3) By placing a circle on coordinate paper so that the radius may be read to tenths, determine the line functions of 200°. Prove that these lines represent the trigonometric functions.

Make up 26, 27, 30 P 37

P 31-33 I a, c, e, g. P 43-4

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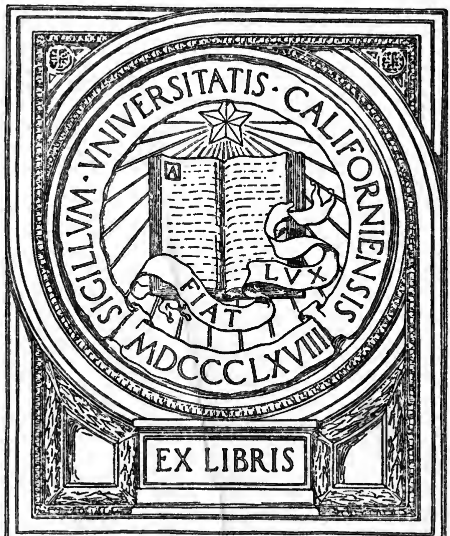
JOHN ALEXANDER JAMESON, Jr.

1903-1934



This book belonged to John Alexander Jameson, Jr., A.B., Williams, 1925; B.S., Massachusetts Institute of Technology, 1928; M.S., California, 1933. He was a member of Phi Beta Kappa, Tau Beta Pi, the American Society of Civil Engineers, and the Sigma Phi Fraternity. His untimely death cut short a promising career. He was engaged, as Research Assistant in Mechanical Engineering, upon the design and construction of the U. S. Tidal Model Laboratory of the University of California.

His genial nature and unostentatious effectiveness were founded on integrity, loyalty, and devotion. These qualities, recognized by everyone, make his life a continuing beneficence. Memory of him will not fail among those who knew him.

PLANE TRIGONOMETRY AND NUMERICAL COMPUTATION

BY

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New York
THE MACMILLAN COMPANY
1919

1, 2 P. 59.
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Set up and electrotyped. Published October, 1919.

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Problems 9 - 17 + 22 P. 84
11 - 17 P. 89
Jan 4 2a, b, 5a, b, 10 all parts 11, 13 P. 92

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

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PREFACE

Ever since the publication of our *Elementary Mathematical Analysis* (The Macmillan Co., 1917) we have been asked by numerous teachers to publish separately, as a textbook in plane trigonometry, the material on trigonometry and logarithms of the text mentioned.

The present textbook is the direct outcome of these requests. Of course, such separate publication of material taken out of the body of another book necessitated some changes and an introductory chapter. As a matter of fact, however, we have found it desirable to make a number of changes and additions not required by the necessities of separate publication. As a result fully half of the material has been entirely rewritten, with the purpose of bringing the text abreast of the most recent tendencies in the teaching of trigonometry.

There is an increasing demand for a brief text emphasizing the numerical aspect of trigonometry and giving only so much of the theory as is necessary for a thorough understanding of the numerical applications. The material has therefore been arranged in such a way that the first six chapters give the essentials of a course in numerical trigonometry and logarithmic computation. The remainder of the theory usually given in the longer courses is contained in the last two chapters.

More emphasis than hitherto has been placed on the use of tables. For this purpose a table of squares and square roots has been added. Recent experience has emphasized the applications of trigonometry in navigation. We have accordingly added some material in the text on navigation, have introduced the haversine, and have added a four-place table of haversines for the benefit of those teachers who feel that the use of the haversine in the solution of triangles is desirable. This material can, however, be readily omitted by any teacher who prefers to do so.

J. W. Young,
F. M. Morgan.

Hanover, N.H.,
August, 1919.

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PLANE TRIGONOMETRY AND NUMERICAL COMPUTATION

CHAPTER I

INTRODUCTORY CONCEPTIONS

1. The Uses of Trigonometry. The word "trigonometry" is derived from two Greek words meaning "the measurement of triangles." A triangle has six so-called *elements* (or *parts*); viz., its three sides and its three angles. We know from our study of geometry that, in general, if three elements of a triangle (not all angles) are given, the triangle is completely determined.* Hence, if three such determining elements of a triangle are given, it should be possible to compute the remaining elements. The methods by which this can be done, *i.e.* methods for "solving a triangle," constitute one of the principal objects of the study of trigonometry.

If two of the angles of a triangle are given, the third angle can be found from the relation $A + B + C = 180°$ (A, B, and C representing the angles of the triangle); also, in a *right* triangle, if two of the sides are known, the third side can be found from the relation $a^2 + b^2 = c^2$ (a, b being the legs and c the hypotenuse). But this is nearly the limit to which the methods of elementary geometry will allow us to go in the solution of a triangle.

Trigonometry † is the foundation of the art of surveying

* What exceptions are there to this statement?

† Throughout this book we shall confine ourselves to the subject of "plane trigonometry," which deals with rectilinear triangles in a plane. "Spherical trigonometry" deals with similar problems regarding triangles on a sphere whose sides are arcs of great circles.

and of much of the art of navigation. It is, moreover, of primary importance in practically every branch of pure and applied mathematics. Many of the more elementary applications will be presented in later portions of this text.

2. The "Shadow Method." The ancient Greeks employed the theory of similar triangles in the solution of a special type of triangle problem which it is worth our while to examine briefly, because it contains the germ of the theory of trigonometry.

It is desired to find the height CA of a vertical tower standing on a level plain. It is observed that at a certain time the tower casts a shadow 42 ft. long. At the same time a pole $C'A'$, 10 ft. long, held vertically with one end on the ground casts a shadow 7 ft. long. From these data the height of the tower is readily computed as follows: The right triangles ABC and $A'B'C'$ are similar since $\angle B = \angle B'$. (Why?) Therefore we have

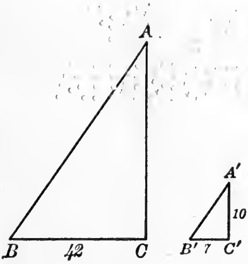


FIG. 1

$$\frac{CA}{BC} = \frac{C'A'}{B'C'} = \frac{10}{7}$$

or

$$CA = \frac{C'A'}{B'C'} \cdot BC = \frac{10}{7} \times 42 = 60.$$

The tower is then 60 ft. high.

3. A "Function" of an Angle. From the point of view of our future study the important thing to notice in the solution of the preceding article is the fact that the ratios $\frac{CA}{BC}$, $\frac{C'A'}{B'C'}$ are equal, *i.e.* that *the ratio of the side opposite the angle B to the side adjacent to the angle is determined by the size of the angle, and does not depend at all on any of the other elements of the triangle*, provided only it is a right triangle.

DEFINITION. Whenever a quantity depends for its value on a second quantity, the first is called a ***function*** of the second.

Thus in our example the ratio of the side opposite an angle of a right triangle to the side adjacent is a quantity which depends for its value only on the angle; it is, therefore, called a *function of the angle.* This ratio is merely one of several functions of an angle which we shall define in the next chapter. By means of these functions the fundamental problem of trigonometry can be readily solved.

The particular function which we have discussed is called the *tangent* of the angle. Explicitly defined for an acute angle of a right triangle, we have

$$\text{tangent of angle} = \frac{\text{side opposite the angle}}{\text{side adjacent to the angle}}.$$

If the angle B in the preceding example were measured it would be found to contain 55°. In any right triangle then containing an angle of 55° we should find this ratio to be equal to $\frac{10}{7}$, or 1.43. If the angle is changed, this ratio is changed, but it is fixed for any given angle. If the angle is 45°, the tangent is equal to 1, since in that case the triangle is isosceles.

The word tangent is abbreviated "tan." Thus we have already found $\tan 55° = 1.43$ and $\tan 45° = 1.00$. Similarly to every other acute angle corresponds a definite number, which is the tangent of that angle. The values of the tangents of angles have been tabulated. We shall have occasion to use such tables extensively in the future.

If a, b, c are the sides of a right triangle ABC with right angle at C and with the usual notation whereby the side a is opposite the angle A and side b opposite the angle B, the definition of the tangent gives

$$\tan B = \frac{b}{a}.$$

From this we get at once,

$$b = a \tan B \quad \text{and} \quad a = \frac{b}{\tan B}.$$

These are our first trigonometric formulas. By means of them and a table of tangents we can compute either leg of a right triangle, if the other leg and an acute angle are given.

EXERCISES

1. What is meant by "the elements of a triangle"? by "solving a triangle"?

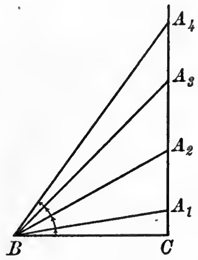
2. A tree casts a shadow 20 ft. long, when a vertical yardstick with one end on the ground casts a shadow of 2 ft. How high is the tree?

3. A chimney is known to be 90 ft. high. How long is its shadow when a 9-foot pole held vertically with one end on the ground casts a shadow 5 ft. long?

4. Give examples from your own experience of quantities which are functions of other quantities.

5. Define the tangent of an acute angle of a right triangle. Why does its value depend only on the size of the angle?

6. In the adjacent figure think of the line BA as rotating about the point B in the direction of the arrow, starting from the position BC (when the angle B is 0) and assuming successively the positions BA_1, BA_2, BA_3, $\cdots$.



Show that the tangent of the angle B is very small when B is very small, that tan B increases as the angle increases, that tan B is less than 1 as long as B is less than 45°, that tan 45° = 1, that tan B is greater than 1 if the angle is greater than 45°, and that tan B increases without limit as B approaches 90°.

7. The following table gives the values of the tangent for certain values of the angle.

angle	10°	20°	30°	40°	50°	60°	70°
tangent	0.176	0.364	0.577	0.839	1.19	1.73	2.75

By means of this table find the other leg of a right triangle ABC from the elements given :

(a) $B = 50°, a = 10$ (d) $B = 20°, b = 13$ (g) $B = 60°, a = 37$
(b) $B = 70°, a = 16$ (e) $A = 30°, b = 5$ (h) $A = 20°, a = 22$
(c) $B = 40°, b = 24$ (f) $A = 10°, b = 62$

8. From the data and the results of the preceding exercise find the other acute angle and the hypotenuse of each of the right triangles.

4. Coördinates in a Plane. The student should already be familiar from his study of algebra with the method of locating points in a plane by means of coördinates. Since we shall often have occasion to use such a method in the future, we will recall it briefly at this point.

The method consists in referring the points in question to two straight lines $X'X$ and $Y'Y$, at right angles to each other, which are called the ***axes of coördinates.*** $X'X$ is usually drawn horizontally and is called the ***x-axis*** ; $Y'Y$, which is then vertical, is called the ***y-axis.***

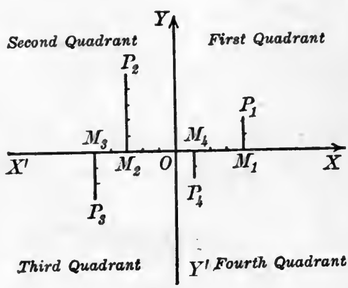


FIG. 2

The position of any point P is completely determined if its distance (measured in terms of some convenient unit) and its direction from each of the axes is known. Thus the position of P_1 (Fig. 2) is known, if we know that it is 4 units to the right of the y-axis and 2 units above the x-axis. If we agree to consider distance measured to the right or upwards as positive, and therefore distance measured to the left or downward as negative; and if, furthermore, we represent distances and directions measured parallel to the x-axis by x, and distances and directions measured parallel to the y-axis by y, then the position of P_1 may be completely given by the specifications $x = +4$, $y = +2$; or more briefly still by the symbol (4, 2).

Similarly, the point P_2 in Fig. 2 is completely determined by the symbol (–3, 5). Observe that in such a symbol the x of the point is written first, the y second. The two numbers x and y, determining the position of a point, are called the ***coördinates*** of the point, the x being called the ***x-coördinate*** or ***abscissa***, the y being called the ***y-coördinate*** or ***ordinate*** of the point. What are the coördinates of P_3 and P_4 in Fig. 2?

The two axes of coördinates divide the plane into four regions called ***quadrants***, numbered as in Fig. 2. The quadrant in which a point lies is completely determined by the signs of its coördinates. Thus points in the first quadrant are characterized by coördinates $(+, +)$, those in the second by $(-, +)$, those in the third by $(-, -)$, and those in the fourth by $(+, -)$.

Square-ruled paper (so-called coördinate or cross section paper) is used to advantage in "plotting" (*i.e.* locating) points by means of their coördinates.

5. Magnitude and Directed Quantities. In the last article we introduced the use of positive and negative numbers, *i.e.* the so-called signed numbers, while in the preceding articles, where we were concerned with the sides and angles of triangles, we dealt only with unsigned numbers. The latter represent magnitude or size only (as a length of 20 ft.), while the former represent both a magnitude and one of two opposite directions or senses (as a distance of 20 ft. to the left of a given line). We are thus led to consider two kinds of quantities: (1) magnitudes, and (2) directed quantities. Examples of the former are: the length of the side of a triangle, the weight of a barrel of flour, the duration of a period of time, etc. Examples of the latter are: the coördinates of a point, the temperature (a certain number of degrees above or below zero), the time at which a certain event occurred (a certain number of hours before or after a given instant), etc.

Geometrically, the distinction between directed quantities and mere magnitudes corresponds to the fact that, on the one hand, we may think of the line segment AB as drawn from A to B or from B to A; and, on the other hand, we may choose to consider only the length of such a segment, irrespective of its direction. Figure 3 exhibits the geometric representation of 5, + 5, and − 5. A segment whose direction is definitely taken account of is called a ***directed segment.*** The magnitude of a directed quantity is called its ***absolute value.*** Thus the absolute value of − 5 (and also of + 5) is 5. Observe that the segments OM_1, M_1P_1 (Fig. 2) representing the coordinates of P_1 are directed segments.

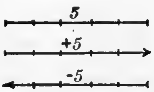


FIG. 3

6. Directed and General Angles. In elementary geometry an angle is usually defined as the figure formed by two half-lines issuing from a point. However, it is often more serviceable to think of an angle as being generated by the rotation in a plane of a half-line OP about the point O as a pivot, starting from the *initial position* OA and ending at the *terminal position* OB (Fig. 4). We then say that the line OP has generated the angle AOB. Similarly, if OP rotates from the initial position OB, to the terminal position OA, then the angle BOA is said to be generated. Considerations similar to those regarding directed line segments (§ 5) lead us to regard one of the above directions of rotation as positive and the other as negative. It is of course quite immaterial which one of the two rotations we regard as positive, but we shall assume, from now on, that *counterclockwise rotation* is *positive* and *clockwise rotation* is *negative.*

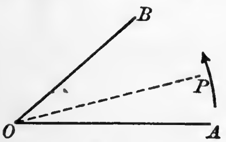


FIG. 4

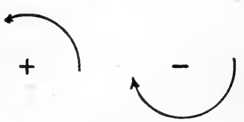


FIG. 5

Still another extension of the notion

of angle is desirable. In elementary geometry no angle greater than 360° is considered and seldom one greater than 180°. But from the definition of an angle just given, we see that the revolving line OP may make any number of complete revolutions before coming to rest, and thus the angle generated may be of any magnitude. Angles generated in this way abound in practice and are known as *angles of rotation.**

When the rotation generating an angle is to be indicated, it is customary to mark the angle by means of an arrow starting at the initial line and ending at the terminal line. Unless some such device is used, confusion is liable to result. In Fig. 6

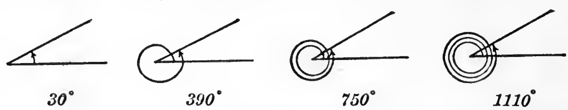


FIG. 6

angles of 30°, 390°, 750°, 1110°, are drawn. If the angles were not marked one might take them all to be angles of 30°.

7. Measurement of Angles. For the present, angles will be measured as in geometry, the degree (°) being the unit of measure. A complete revolution is 360°. The other units in this system are the minute (′), of which 60 make a degree, and the second (″), of which 60 make a minute. This system of units is of great antiquity, having been used by the Babylonians. The considerations of the previous article then make it clear that any real number, positive or negative, may represent an angle, the absolute value of the number representing the magnitude of the angle, the sign representing the direction of rotation.

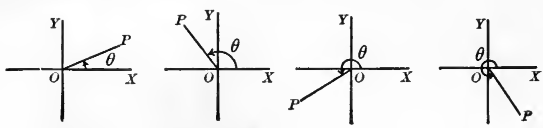


FIG. 7

Consider the angle $XOP = \theta$, whose vertex O coincides with the origin O of a system of rectangular coördinates, and whose initial line OX coin-

*For example, the minute hand of a clock describes an angle of −180° n 30 minutes, an angle of −540° in 90 minutes, and an angle of −720° in 120 ninutes.

cides with the positive half of the x-axis (Fig. 7). The angle θ is then said to be in the first, second, third, or fourth quadrant, according as its terminal line OP is in the first, second, third, or fourth quadrant.

8. Addition and Subtraction of Directed Angles. The meaning to be attached to the sum of two directed angles is analogous to that for the sum of two directed line segments. Let a and b be two half-lines issuing from the same point O and let (ab) represent an angle obtained by rotating a half-line from the position a to the position b. Then if we have two angles (ab) and (bc) with the same vertex O, the sum $(ab)+(bc)$ of the angles is the angle represented by the rotation of a half-line from the position a to the position b and then rotating from the position b to the position c. But these two rotations are together equivalent to a single rotation from a to c, no matter what the relative positions of a, b, c may have been. Hence, we have for any three half-lines a, b, c issuing from a point O,

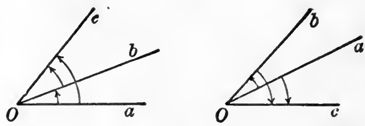


FIG. 8

(1) $\quad (ab)+(bc)=(ac), \quad (ab)+(ba)=0, \quad (ab)=(cb)-(ca).$

It must be noted, however, that the equality sign here means "equal, *except possibly for multiples of 360°*." The proof of the last relation is left as an exercise.

EXERCISES

1. On square-ruled paper draw two axes of reference and then plot the following points: $(2, 3)$, $(-4, 2)$, $(-7, -1)$, $(0, -3)$, $(2, -5)$, $(5, 0)$.

2. What are the coördinates of the origin?

3. Where are all the points for which $x=2$? $x=-3$? $y=-1$? $y=4$? $x=0$?

4. Show that any point P on the y-axis has coördinates of the form $(0, y)$. What is the form of the coördinates of any point on the x-axis?

5. A right triangle has the vertex of one acute angle at the origin and one leg along the x-axis. The vertex of the other acute angle is at $(7, 10)$. What is the tangent of the angle at O?

6. What angle does the minute hand of a clock describe in 2 hours and 30 minutes? in 4 hours and 20 minutes?

7. Suppose that the dial of a clock is transparent so that it may be read from both sides. Two persons stationed at opposite sides of the dial observe the motion of the minute hand. In what respect will the angles described by the minute hand as seen by the two persons differ?

8. In what quadrants are the following angles: 87°? 135°? −325°? 540°? 1500°? −270°?

9. In what quadrant is $\theta/2$ if θ is a positive angle less than 360° and in the second quadrant? third quadrant? fourth quadrant?

10. By means of a protractor construct $27° + 85° + (-30°) + 20° + (-45°)$.

11. By means of a protractor construct $-130° + 56° - 24°$.

CHAPTER II

THE RIGHT TRIANGLE

9. Introduction. At the beginning of the preceding chapter we described the fundamental problem of trigonometry to be the "solution of the triangle," *i.e.* the problem of computing the unknown elements of a triangle when three of the elements (not all angles) are given. This problem can be solved by finding relations between the sides and angles of a triangle by means of which it is possible to express the unknown elements in terms of the known elements. In order to establish such relations, it has been found desirable to define certain functions of an angle. One such function — the tangent — was introduced in § 3 by way of preliminary illustration.

In the present chapter, we shall give a new definition of the tangent of an angle and also define two other equally important functions — the *sine* and the *cosine*. It should be noted that the definition given for the tangent in § 3 applies only to an acute angle of a right triangle. For the purposes of a systematic study of trigonometry we require a more general definition, which will apply to any angle, positive or negative, and of any magnitude. Such definitions are given in the next article, in which the notion of a system of coördinates plays a fundamental rôle, the notion of a triangle not being introduced at all. After considering some of the consequences of our definitions in §§ 11–13, we consider the way in which these definitions enable us to express relations between the sides and angles of a right triangle. These results are then immediately applied to the solution of numerical problems by means of tables and to applications in surveying and navigation.

10. The Sine, Cosine, and Tangent of an Angle. We may now define three of the functions referred to in § 3. To this end let $\theta = XOP$ (Fig. 9) be any directed angle, and let

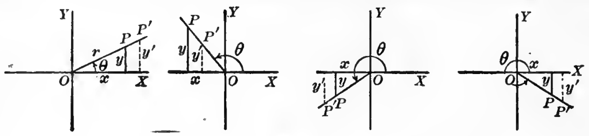


FIG. 9

us establish a system of rectangular coördinates in the plane of the angle such that the initial side OX of the angle is the positive half of the x-axis, the vertex O being at the origin and the y-axis being in the usual position with respect to the x-axis. Let the units on the two axes be equal. Finally, let P be any point other than O on the terminal side of the angle θ, and let its coördinates be (x, y). The directed segment $OP = r$ is called the *distance of P* and is always chosen positive. The coördinates x and y are positive or negative according to the conventions previously adopted. We then define

$$\textit{The sine of } \theta = \frac{\text{ordinate of } P}{\text{distance of } P} = \frac{y}{r},$$

$$\textit{The cosine of } \theta = \frac{\text{abscissa of } P}{\text{distance of } P} = \frac{x}{r},$$

$$\textit{The tangent of } \theta = \frac{\text{ordinate of } P}{\text{abscissa of } P} = \frac{y}{x}, \text{ provided } x \neq 0.*$$

These functions are usually written in the abbreviated forms $\sin \theta$, $\cos \theta$, $\tan \theta$, respectively; but they are read as "sine θ," "cosine θ," "tangent θ." It is very important to notice that *the values of these functions are independent of the position of the point P on the terminal line.* For let $P'(x', y')$ be any other point on this line. Then from the similar right triangles xyr † and $x'y'r'$ it follows that the ratio of any two sides of the triangle xyr is equal in magnitude and sign to the

* Prove that x and y cannot be zero simultaneously.

† Triangle xyz means the triangle whose sides are x, y, z.

ratio of the corresponding sides of the triangle $x'y'r'$. Therefore the values of the functions just defined depend merely on the angle θ. They are one-valued functions of θ and are called ***trigonometric functions.***

Since the values of these functions are defined as the ratios of two directed segments, they are abstract numbers. They may be either positive, negative, or zero. Remembering that r is always positive, we may readily verify that the signs of the three functions are given by the following table.

Quadrant	1	2	3	4
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−

11. Values of the Functions for 45°, 135°, 225°, 315°. In each of these cases the triangle xyr is isosceles. Why? Since the trigonometric functions are independent of the position of the point P on the terminal line, we may choose the legs of the right triangle xyr to be of length unity, which

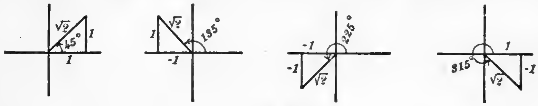


FIG. 10

gives the distance OP as $\sqrt{2}$. Figure 10 shows the four angles with all lengths and directions marked. Therefore,

$$\sin 45° = \frac{1}{\sqrt{2}}, \qquad \cos 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1,$$

$$\sin 135° = \frac{1}{\sqrt{2}}, \qquad \cos 135° = -\frac{1}{\sqrt{2}}, \qquad \tan 135° = -1,$$

$$\sin 225° = -\frac{1}{\sqrt{2}}, \qquad \cos 225° = -\frac{1}{\sqrt{2}}, \qquad \tan 225° = 1,$$

$$\sin 315° = -\frac{1}{\sqrt{2}}, \qquad \cos 315° = \frac{1}{\sqrt{2}}, \qquad \tan 315° = -1.$$

12. Values of the Functions for 30°, 150°, 210°, 330°. From geometry we know that if one angle of a right triangle contains 30°, then the hypotenuse is double the shorter leg, which is opposite the 30° angle. Hence if we choose the shorter leg (ordinate) as 1, the hypotenuse (distance) is 2,

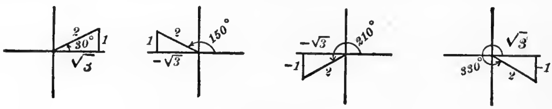


FIG. 11

and the other leg (abscissa) is $\sqrt{3}$. Figure 11 shows angles of 30°, 150°, 210°, 330° with all lengths and directions marked. Hence we have

$\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 30° = \frac{1}{\sqrt{3}}$,

$\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$, $\tan 150° = -\frac{1}{\sqrt{3}}$,

$\sin 210° = -\frac{1}{2}$, $\cos 210° = -\frac{\sqrt{3}}{2}$, $\tan 210° = \frac{1}{\sqrt{3}}$,

$\sin 330° = -\frac{1}{2}$, $\cos 330° = \frac{\sqrt{3}}{2}$, $\tan 330° = -\frac{1}{\sqrt{3}}$.

13. Values of the Functions for 60°, 120°, 240°, 300°. It is left as an exercise to construct these angles and to prove that

$\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, $\tan 60° = \sqrt{3}$,

$\sin 120° = \frac{\sqrt{3}}{2}$, $\cos 120° = -\frac{1}{2}$, $\tan 120° = -\sqrt{3}$,

$\sin 240° = -\frac{\sqrt{3}}{2}$, $\cos 240° = -\frac{1}{2}$, $\tan 240° = \sqrt{3}$,

$\sin 300° = -\frac{\sqrt{3}}{2}$, $\cos 300° = \frac{1}{2}$, $\tan 300° = -\sqrt{3}$.

14. Sides and Angles of a Right Triangle. Evidently any right triangle ABC can be so placed in a system of coördinates that the vertex of either acute angle coincides with the origin O and that the adjacent leg lies along the positive end OX of the x-axis (Fig. 12). The following relations then follow at once from the definitions of the sine, cosine, and tangent of § 10.

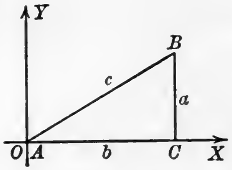


FIG. 12

In any right triangle, the trigonometric functions of either acute angle are given by the ratios:

$$\textit{the sine} = \frac{\textit{side opposite the angle}}{\textit{hypotenuse}},$$

$$\textit{the cosine} = \frac{\textit{side adjacent to the angle}}{\textit{hypotenuse}},$$

$$\textit{the tangent} = \frac{\textit{side opposite the angle}}{\textit{side adjacent to the angle}}.$$

These relations are fundamental in all that follows. They should be firmly fixed in mind in such a way that they can be readily applied to any right triangle in whatever position it may happen to be (for example as in Fig. 13). The student should be able to reproduce any of the following relations without hesitation whenever called for. They should not be memorized, but should be read from an actual or imagined figure:

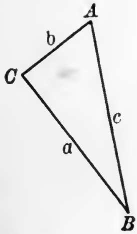


FIG. 13

$$\sin A = \frac{a}{c}, \quad \sin B = \frac{b}{c},$$

$$\cos A = \frac{b}{c}, \quad \cos B = \frac{a}{c},$$

$$\tan A = \frac{a}{b}, \quad \tan B = \frac{b}{a}.$$

Also the known relation: $c^2 = a^2 + b^2.$

If any two elements (other than the right angle) of a right triangle are given, we can then find a relation connecting these two elements with any unknown element, from which relation the unknown element can be computed.

15. Applications. The angle which a line from the eye to an object makes with a horizontal line in the same vertical plane is called an *angle of elevation* or an *angle of depression*,

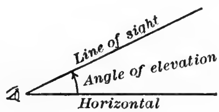
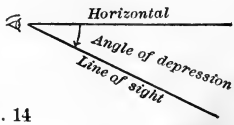


FIG. 14



according as the object is above or below the eye of the observer (Fig. 14). Such angles occur in many examples.

EXAMPLE 1. A man wishing to know the distance between two points A and B on opposite sides of a pond locates a point C on the land (Fig. 15) such that $AC = 200$ rd., angle $C = 30°$, and angle $B = 90°$. Find the distance AB.

SOLUTION :

$$\frac{AB}{AC} = \sin C. \qquad \text{(Why ?)}$$

$$AB = AC \cdot \sin C$$
$$= 200 \cdot \sin 30°$$
$$= 200 \cdot \tfrac{1}{2} = 100 \text{ rd.}$$

FIG. 15

EXAMPLE 2. Two men stationed at points A and C 800 yd. apart and in the same vertical plane with a balloon B, observe simultaneously the angles of elevation of the balloon to be 30° and 45° respectively. Find the height of the balloon.

SOLUTION : Denote the height of the balloon DB by y, and let $DC = x$; then $AD = 800 - x$.

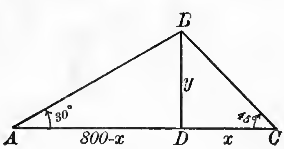


FIG. 16

Since $\tan 45° = 1$, we have $1 = \frac{y}{x}$,

and since $\tan 30° = 1/\sqrt{3}$, we have $\frac{1}{\sqrt{3}} = \frac{y}{800 - x}$.

Therefore $x = y$ and $800 - x = y\sqrt{3}$.

Solving these equations for y, we have $y = \frac{800}{\sqrt{3} + 1} = 292.8$ yd.

EXERCISES

1. In what quadrants is the sine positive? cosine negative? tangent positive? cosine positive? tangent negative? sine negative?

2. In what quadrant does an angle lie if

(*a*) its sine is positive and its cosine is negative?

(*b*) its tangent is negative and its cosine is positive?

(*c*) its sine is negative and its cosine is positive?

(*d*) its cosine is positive and its tangent is positive?

3. Which of the following is the greater and why: $\sin 49°$ or $\cos 49°$? $\sin 35°$ or $\cos 35°$?

4. If θ is situated between 0° and 360°, how many degrees are there in θ if $\tan \theta = 1$? Answer the similar question for $\sin \theta = \frac{1}{2}$; $\tan \theta = -1$.

5. Does $\sin 60° = 2 \cdot \sin 30°$? Does $\tan 60° = 2 \cdot \tan 30°$? What can you say about the truth of the equality $\sin 2\theta = 2 \sin \theta$?

6. The Washington Monument is 555 ft. high. At a certain place in the plane of its base, the angle of elevation of the top is 60°. How far is that place from the foot and from the top of the tower?

7. A boy whose eyes are 5 ft. from the ground stands 200 ft. from a flagstaff. From his eyes, the angle of elevation of the top is 30°. How high is the flagstaff?

8. A tree 38 ft. high casts a shadow 38 ft. long. What is the angle of elevation of the top of the tree as seen from the end of the shadow? How far is it from the end of the shadow to the top of the tree?

9. From the top of a tower 100 ft. high, the angle of depression of two stones, which are in a direction due east and in the plane of the base are 45° and 30° respectively. How far apart are the stones?

Ans. $100(\sqrt{3} - 1) = 73.2$ ft.

10. Find the area of the isosceles triangle in which the equal sides 10 inches in length include an angle of 120°. *Ans.* $25\sqrt{3} = 43.3$ sq. in.

11. Is the formula $\sin 2\theta = 2 \sin \theta \cos \theta$ true when $\theta = 30°$? 60°? 120°?

12. From a figure prove that $\sin 117° = \cos 27°$.

13. Determine whether each of the following formulas is true when $\theta = 30°, 60°, 150°, 210°$:

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta},$$

$$1 + \frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta},$$

$$\sin^2\theta + \cos^2\theta = 1.$$

14. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points the distance between which is r (the units on the axes being equal). If θ is the angle that the line P_1P_2 makes with the x-axis, prove that

$$\frac{x_2 - x_1}{\cos\theta} + \frac{y_2 - y_1}{\sin\theta} = 2r.$$

16. Computation of the Value of One Trigonometric Function from that of Another.

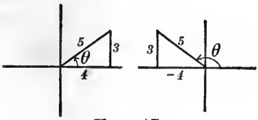


FIG. 17

EXAMPLE 1. Given that $\sin\theta = \frac{3}{5}$, find the values of the other functions.

Since $\sin\theta$ is positive, it follows that θ is an angle in the first or in the second quadrant. Moreover, since the value of the sine is $\frac{3}{5}$, then $y = 3 \cdot k$ and $r = 5 \cdot k$, where k is any positive constant different from zero. (Why?) It is, of course, immaterial what positive value we assign to k, so we shall assign the value 1. We know, however, that the abscissa, ordinate, and distance are connected by the relation $x^2 + y^2 = r^2$, and hence it follows that $x = \pm 4$. Figure 17 is then self-explanatory. Hence we have, for the first quadrant, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, and $\tan\theta = \frac{3}{4}$; for the second quadrant, $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$.

EXAMPLE 2. Given that $\sin\theta = \frac{5}{13}$ and that $\tan\theta$ is negative, find the other trigonometric functions of the angle θ.

Since $\sin\theta$ is positive and $\tan\theta$ is negative, θ must be in the second quadrant. We can, therefore, construct the angle (Fig. 18), and we obtain $\sin\theta = \frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = -\frac{5}{12}$.

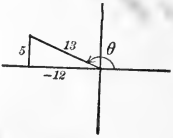


FIG. 18

17. Computation for Any Angle. Tables. The values of the trigonometric functions of any angle may be computed by the graphic method. For example, let us find the trigonometric functions of 35°. We first construct on square-ruled paper, by means of a protractor, an angle of 35° and choose a point P on the terminal line so that OP shall equal 100 units. Then from the figure we find that $OM = 82$ units and $MP = 57$ units. Therefore

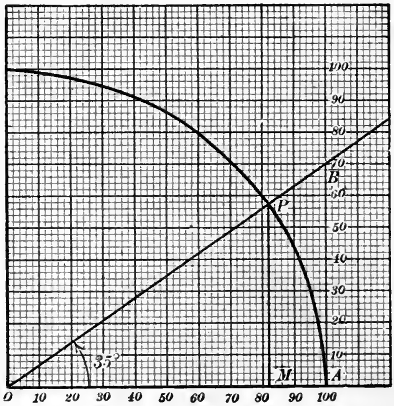


FIG. 19

$$\sin 35° = \tfrac{57}{100} = 0.57,\ \cos 35° = \tfrac{82}{100} = 0.82,\ \tan 35° = \tfrac{57}{82} = 0.70.$$

The tangent may be found more readily if we start by taking $OA = 100$ units and then measure AB. In this case, $AB = 70$ units and hence $\tan 35° = \tfrac{70}{100} = 0.70$.

It is at once evident that the graphic method, although simple, gives only an approximate result. However, the values of these functions have been computed accurately by methods beyond the scope of this book. The results have been put in tabular form and are known as tables of natural trigonometric functions. Such tables and how to use them will be discussed in the next article.

Figure 20 makes it possible to read off the sine, cosine, or tangent of any angle between 0° and 90° with a fair degree of accuracy. The figure is self-explanatory. In reading off values of the tangent use the vertical line through 100 for angles up to 55°, and the line through 10 for angles greater than 55°. Its use is illustrated in some of the following exercises.

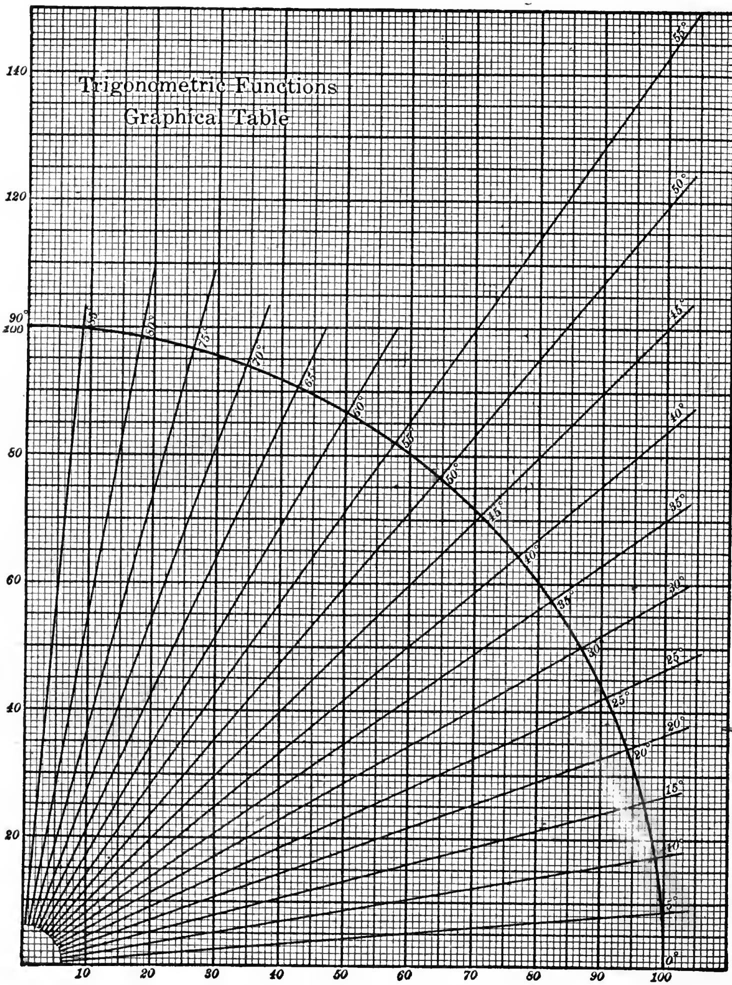


FIG. 20. — GRAPHICAL TABLE OF TRIGONOMETRIC FUNCTIONS

EXERCISES

Find the other trigonometric functions of the angle θ when

1. $\tan\theta = -3$.
2. $\sin\theta = -\frac{3}{5}$.
3. $\cos\theta = \frac{12}{13}$.
4. $\tan\theta = \frac{4}{7}$.
5. $\sin\theta = \frac{3}{4}$.
6. $\cos\theta = -\frac{1}{5}$.
7. $\sin\theta = \frac{3}{5}$ and $\cos\theta$ is negative.
8. $\tan\theta = 2$ and $\sin\theta$ is negative.
9. $\sin\theta = -\frac{1}{4}$ and $\tan\theta$ is positive.
10. $\cos\theta = \frac{3}{5}$ and $\tan\theta$ is negative.
11. Can 0.6 and 0.8 be the sine and cosine, respectively, of one and the same angle? Can 0.5 and 0.9? *Ans.* Yes; no.
12. Is there an angle whose sine is 2? Explain.
13. Determine graphically the functions of 20°, 38°, 70°, 110°.
14. From Fig. 20, find values of the following:
    sin 10°, cos 50°, tan 40°, sin 80°, tan 70°, cos 32°, tan 14°, sin 14°.
15. A tower stands on the shore of a river 200 ft. wide. The angle of elevation of the top of the tower from the point on the other shore exactly opposite to the tower is such that its sine is $\frac{2}{5}$. Find the height of the tower.
16. From a ship's masthead 160 feet above the water the angle of depression of a boat is such that the tangent of this angle is $\frac{1}{12}$. Find the distance from the boat to the ship. *Ans.* 640 yards.

18. Use of Tables of Trigonometric Functions. Examination of the tables of "Four Place Trigonometric Functions" (p. 112) shows columns headed "Degrees," "Sine," "Tangent," "Cosine," and under each of the last three named a column headed "Value" (none of the other columns concern us at present). Two problems regarding the use of these tables now present themselves.

1. *To find the value of a function when the angle is given.*

(*a*) Find the value of sin 15° 20′. In the column headed "Degrees" locate the line corresponding to 15° 20′ (p. 113); on the same line in the "value" column for the "Sine," we read the result: sin 15° 20′ = 0.2644. On the same line, by using the proper column, we find tan 15° 20′ = 0.2742, and cos 15° 20′ = 0.9644.

(*b*) Find the value of tan 57° 50′. The entries in the column marked "Degrees" at the *top* only go as far as 45° (p. 116). But the columns marked "Degrees" at the *bottom* contain entries beginning with 45° (p. 116) and running backwards to 90° (p. 112). In using these entries we must use the designations at the bottom of the columns. Thus on the line corresponding to 57° 50′ (p. 115) we find the desired value: tan 57° 50′ = 1.5900. Also sin 57° 50′ = 0.8465, and cos 57° 50′ = 0.5324.

(*c*) Find the value of sin 34° 13′. This value lies between the values of sin 34° 10′ and 34° 20′. We find for the latter

$$\begin{aligned} \sin 34° \, 10' &= 0.5616 \\ \sin 34° \, 20' &= \underline{0.5640} \\ \text{Difference for } 10' &= 0.0024 \end{aligned}$$

Assuming that the change in the value of the function throughout this small interval is proportional to the change in the value of the angle, we conclude that the change for 1′ in the angle would be 0.00024. For 3′, the change in the value of the function would then be 0.00072. Neglecting the 2 in the last place (since we only use four places and the 2 is less than 5), we find sin 34° 13′ = 0.5616 + 0.0007 = 0.5623. This process is called ***interpolation.*** With a little practice all the work involved can and should be done mentally; *i.e.* after locating the place in the table (and marking it with a finger), we observe that the "tabular difference" is "24"; we calculate mentally that .3 of 24 is 7.2, and then add 7 to 5616 as we write down the desired value 0.5623.

Similarly we find tan 34° 13′ = 0.6800 (the correction to be added is in this case 12.9 which is "rounded off" to 13) and cos 34° 13′ = 0.8269. (Observe that in this case the correction must be subtracted. Why?)

2. *To find the angle when a value of a function is given.* Here we proceed in the opposite direction. Given sin A =

0.3289; find A. An examination of the sine column shows that the given value lies between sin 19° 10′ (= 0.3283) and sin 19° 20′(= 0.3311). We note the tabular difference to be 28. The correction to be applied to 19° 10′ is then $\frac{6}{28}$ of 10′ = $\frac{60'}{28}$ = $\frac{15'}{7}$ = 2.1′. Hence A = 19° 12.1′. (With a four place table do not carry your interpolation farther than the nearest tenth of a minute.) (See § 20.)

EXERCISES

1. For practice in the use of tables, verify the following:

(*a*) sin 18° 20′ = 0.3145	(*d*) sin 27° 14′ = 0.4576	(*g*) sin 62° 24′.1 = 0.8862
(*b*) cos 37° 30′ = 0.7934	(*e*) cos 34° 11′ = 0.8272	(*h*) cos 59° 46′.2 = 0.5034
(*c*) tan 75° 50′ = 3.9617	(*f*) tan 68° 21′ = 2.5173	(*i*) tan 14° 55′.6 = 0.2665

Assume first that the angles are given and verify the values of the functions. Then assume the values of the functions to be given and verify the angles.

2. A certain railroad rises 6 inches for every 10 feet of track. What angle does the track make with the horizontal?

3. On opposite shores of a lake are two flagstaffs A and B. Perpendicular to the line AB and along one shore, a line BC = 1200 ft. is measured. The angle ACB is observed to be 40° 20′. Find the distance between the two flagstaffs.

4. The angle of ascent of a road is 8°. If a man walks a mile up the road, how many feet has he risen?

5. How far from the foot of a tower 150 feet high must an observer, 6 ft. high, stand so that the angle of elevation of its top may be 23°.5?

6. From the top of a tower the angle of depression of a stone in the plane of the base is 40° 20′. What is the angle of depression of the stone from a point halfway down the tower?

7. The altitude of an isosceles triangle is 24 feet and each of the equal angles contains 40° 20′. Find the lengths of the sides and area of the triangle.

8. A flagstaff 21 feet high stands on the top of a cliff. From a point on the level with the base of the cliff, the angles of elevation of the top and bottom of the flagstaff are observed. Denoting these angles by α and β respectively, find the height of the cliff in case $\sin\alpha = \frac{8}{17}$ and $\cos\beta = \frac{12}{13}$. *Ans.* 75 feet.

9. A man wishes to find the height of a tower CB which stands on a horizontal plane. From a point A on this plane he finds the angle of elevation of the top to be such that $\sin CAB = \frac{3}{5}$. From a point A' which is on the line AC and 100 feet nearer the tower, he finds the angle of elevation of the top to be such that $\tan CA'B' = \frac{3}{2}$. Find the height of the tower.

10. Find the radius of the inscribed and circumscribed circle of a regular pentagon whose side is 14 feet.

11. If a chord of a circle is two thirds of the radius, how large an angle at the center does the chord subtend ?

19. Computation with Approximate Data. Significant Figures. The numerical applications of trigonometry (in surveying, navigation, engineering, etc.) are concerned with computing the values of certain unknown quantities (distances, angles, etc.) from known data which are secured by *measurement*. Now, any direct measurement is necessarily an approximation. A measurement may be made with greater or less accuracy according to the needs of the problem in hand — but it can never be absolutely exact. Thus, the information on a signpost that a certain village is 6 miles distant merely means that the distance is 6 miles *to the nearest mile* — *i.e.* that the distance is between $5\frac{1}{2}$ and $6\frac{1}{2}$ miles. Measurements in a physical or engineering laboratory need sometimes to be made to the nearest one thousandth of an inch. For example the bore of an engine cylinder may be measured to be 3.496 in., which means that the bore is between 3.4955 in. and 3.4965 in.

A simple convention makes it possible to recognize at a glance the degree of accuracy implied by a number representing an approximate measure (either direct or computed). This convention consists simply in the agreement to write no more figures than the accuracy warrants. Thus in arithmetic 6 and 6.0 and 6.00 all mean the same thing. This is not so, when these numbers are used to express the result of measurement or the result of computation from approximate data. Thus 6 means that the result is accurate to the nearest unit, 6.0 that it is accurate to the nearest tenth of a unit, 6.00 to the nearest hundredth of a unit.

These considerations have an important bearing on practical computation. If the side of a square is measured and found to be 3.6 in. and the length of the diagonal is computed by the formula: diagonal = side $\times \sqrt{2}$, it would be wrong to write $= 3.6 \times \sqrt{2} = 3.6 \times 1.4142 = 5.09112$ in. The correct result is 5.1 in. For the computed value of the diagonal cannot be more accurate than the measured value of the side. The result 5.09112 must therefore be "rounded off" to two significant figures, which gives 5.1. As a matter of fact for the purpose of this problem $\sqrt{2} = 1.4142$ should be rounded before multiplication to $\sqrt{2} = 1.4$; thereby reducing the amount of labor necessary.

A number is "**rounded off**," by dropping one or more digits at the right and, if the last digit dropped is 5^+, 6, 7, 8, or 9 increasing the preceding digit by 1.* Thus the successive approximations to π obtained by rounding of 3.14159 ... are 3.1416, 3.142, 3.14, 3.1, 3.

20. The Number of Significant Figures of a number (in the decimal notation) may now be defined as the total number of digits in the number, except that if the number has no digits to the right of the decimal point, any zeros occurring between the decimal point and the first digit different from zero are not counted as significant. Thus, 34.06 and 3,406,000 are both numbers of four significant figures: while 3,406,000.0 is a number of eight significant figures.†

* In rounding off a 5 computers round off to an *even* digit. Thus 1.415 would be rounded to 1.42, whereas 1.445 would be rounded to 1.44. If this rule is used consistently the errors made will tend to compensate each other.

† Confusion will arise in only one case. For example, if 3999.7 were rounded by dropping the 7 we should write it as 4000 which according to the above definition would have only 1 significant figure, whereas we know from the way it was obtained that all four figures are significant. In such a case we may underscore the zeros to indicate they are significant or use some other device.

In any computation involving multiplication or division the number of significant figures is generally used as a measure of the accuracy of the data. A computed result should not in general contain more significant figures than the least accurate of the data. But computers generally retain one additional figure during the computation and then properly round off the final result. Even then the last digit may be inaccurate — but that is unavoidable.

The following general rules will be of use in determining the degree of accuracy to be expected and in avoiding useless labor:

1. Distances expressed to two significant figures call for angles expressed to the nearest 30′ and *vice versa.*

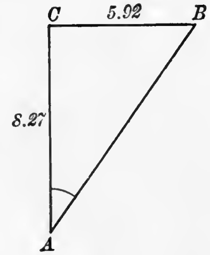
2. Distances expressed to three significant figures call for angles expressed to the nearest 5′, and *vice versa.*

3. Distances expressed to four significant figures call for angles expressed to the nearest minute, and *vice versa.*

4. Distances expressed to five significant figures call for angles expressed to the nearest tenth of a minute, and *vice versa.*

In working numerical problems the student should use every safeguard to avoid errors. Neatness and systematic arrangement of the work are important in this connection. All work should be checked in one or more of the following ways. 1. Gross errors may be detected by habitually asking oneself: Is this result reasonable or sensible? 2. A figure drawn to scale makes it possible to measure the unknown parts and to compare the results of such measurements with the computed results. 3. An accurate check can often be secured with comparatively little additional labor by computing one of the quantities from two different formulas or by verifying a known relation. For example, if the legs a, b of a right triangle have been computed by the formulas $a = c \sin A$ and $b = c \cos A$, we may check by verifying the relation $a^2 + b^2 = c^2$.

EXAMPLE. A straight road is to be built from a point A to a point B which is 5.92 miles east and 8.27 miles north of A. What will be the direction of the road and its length?



Formulas: $\tan A = \dfrac{5.92}{8.27}$; $AB = \dfrac{8.27}{\cos A}$.

Therefore $\tan A = 0.716$ and $A = 35° 35'$,
$\cos A = 0.813$ $AB = 10.17.$*

Check by $a^2 + b^2 = c^2$.

From a table of squares (p. 107, see § 21)

$$(5.92)^2 = 35.05$$
$$(8.27)^2 = 68.39 \qquad (10.17)^2 = 103.4.$$
$$103.4$$

21. Use of Table of Squares. Square Roots. The table of squares of numbers (p. 106) may be used to facilitate computation. In the example of the last article, we required the square of 5.92. We find 5.9 on p. 107 in the left-hand column and find the third digit 2 at the head of a certain column. At the intersection of the line and column thus determined we find the desired result $(5.92)^2 = 35.05$. The square of 8.27 is found similarly at the intersection of the line corresponding to 8.2 and the column headed 7. To find $(10.17)^2$, we find the line corresponding to 1.0 (the first two digits, neglecting the decimal point) and find $(1.01)^2 = 1.020$ and $(1.02)^2 = 1.040$. By interpolating, as explained in § 18, we find $(1.017)^2 = 1.034$. Now shifting the decimal point one place in the "number" requires a corresponding shift of two places in the square. Hence, $(10.17)^2 = 103.4$.

The table can also be used to find the square root of a number. Thus to find $\sqrt{2}$ we find, on working backwards in this table, that 2 lies between 1.988 $[=(1.41)^2]$ and 2.016 $[=(1.42)^2]$. By interpolation we then find $\sqrt{2} = 1.414$, correct to four significant places. [Tabular difference $= 28$; correction $= \frac{120}{28} = 4$ in the fourth place.]

* The retention of four significant figures in AB is justified because the number is so small at the left.

EXERCISES

1. From an observing station 357 ft. above the water, the angle of depression of a ship is $2° 15'$. Find the horizontal distance to the ship in yards.

2. A projectile falls in a straight line making an angle of 25° with the horizontal. Will it strike the top of a tree 24 meters high which is 72 meters from the point where the projectile would strike the ground?

3. At a point 372 ft. from the foot of a cliff surmounted by an observation tower the angle of elevation of the top of the tower is $51° 25'$, and of the foot of the tower $31° 55'$. Find the height of the cliff and of the tower.

4. How far from the foot of a flagpole 130 ft. high must an observer stand so that the angle of elevation of the top of the pole will be 25°?

5. GA is a horizontal line, T is a point vertically above A; B a point vertically below A. The angle BGA in minutes is $\frac{AG}{4000}$. Find $\angle BGT$ in degrees and minutes, given $GA = 10{,}340$ meters; $AT = 416.4$ meters.

6. It is desired to find the height of a wireless tower situated on the top of a hill. The angle subtended by the tower at a point 250 ft. below the base of the tower and at a distance measured horizontally of 2830 ft. from it is found to be $2° 42'$. Find the height of the tower.

7. From a tower 428.3 ft. high the angles of depression of two objects situated in the same horizontal line with the base of the tower and on the same side are $30° 22'$ and $47° 37'$. Find the distance between them.

8. The summit of a mountain known to be 13,260 ft. high is seen at an angle of elevation of $27° 12'$ from a camp located at an altitude of 6359 ft. Compute the air-line distance from the camp to the summit of the mountain.

9. Two towns A and B, of which B is 25 miles northeast of A, are to be connected by a new road. 11 miles of the road is constructed from A in the direction N. 21° E.; what must be length and direction of the remainder of the road, assuming it to be straight?

22. Applications in Navigation. We shall confine ourselves to problems in *plane sailing*; *i.e.* we shall assume that the distances considered are sufficiently small so that the curvature of the earth may be neglected.

DEFINITION. The ***course*** of a ship is the direction in which she is sailing. It is given either by the points of a mariner's compass (Fig. 21) as N. E. by N. or in degrees and minutes *measured clockwise from the north.* Observe that a "point" on a mariner's compass is 11° 15′. Hence for example, the course of a ship could be given either as N. E. by N. or as 33° 45′. A course S. E. by S. is the same as a course of 146° 15′.

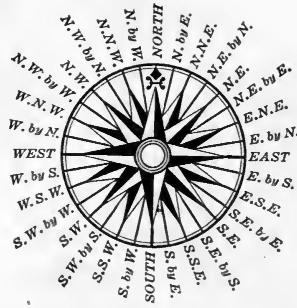


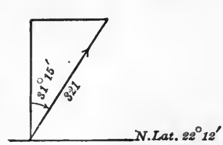
FIG. 21

The ***distance*** a ship travels on a given course is always given in nautical miles or knots. A knot is the length of a minute of arc on the earth's equator. (The earth's circumference is then $360 \times 60 = 21{,}600$ knots.) The horizontal component of the distance is called the ***departure***, the vertical component is called the ***difference in latitude.*** The departure is usually given in miles (knots), the difference in latitude in degrees and minutes.

FIG. 22

EXAMPLE. A ship starts from a position in 22° 12′ N. latitude, and sails 321 knots on a course of 31° 15′. Find the difference in latitude, the departure, and the latitude of the new position of the ship.

$$\begin{aligned}\text{diff. in lat.} &= \text{distance times cosine of course}\\ &= 321 \cos 31° 15'\\ &= 321 \times 0.855 = 274' = 4° 34'.\\ \text{departure} &= \text{distance times sine of course}\\ &= 321 \sin 31° 15'\\ &= 321 \times 0.519 = 167 \text{ knots.}\end{aligned}$$



Since the ship is sailing on a course which increases the latitude, the latitude of the new position is $22° 12' + 4° 34' = 26° 46'$ N.

Knowing the difference in latitude and the departure, we are able to calculate the new position of the ship, if the original position is known. In the preceding example, we found the latitude of the new position from the difference in latitude. To find the difference in longitude from the departure is not quite so simple. As the latitude increases, a given departure implies an increasing difference in longitude. Only on the equator is the departure of one nautical mile equivalent to a difference in longitude of one minute.

The adjacent figure shows a departure AB in latitude ϕ. The difference in longitude (in minutes) corresponding to AB is clearly the number of nautical miles in CD. Now arcs AB and CD are proportional to their radii PA and OC. Or,

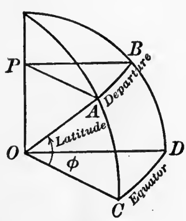


FIG. 23

$$CD = \frac{OC}{PA} \cdot AB = \frac{AB}{\cos \phi}. \quad \text{(Why ?)}$$

In practice, it is customary to take for ϕ in the determination of difference in longitude the so-called ***middle latitude,*** *i.e.* the latitude halfway between the original latitude and the final latitude.

Thus in the preceding example, the original latitude was $22° 12'$ N, the final latitude was $26° 46'$ N. The middle latitude is therefore $\frac{1}{2}(22° 12' + 26° 46') = 24° 29'$. Hence

$$\text{difference in longitude} = \frac{\text{departure}}{\text{cosine of middle latitude}}$$

$$= \frac{167}{\cos 24° 29'} = \frac{167}{0.910} = 184' = 3° 4'.$$

The determination of the position of a ship from its course and distance is known as ***dead reckoning.*** It is subject to considerable inaccuracy and must often in practice be checked by direct determination of position by observations on the sun or stars.

EXERCISES

1. A ship sails N. E. by E. at the rate of 12 knots per hour. Find the rate at which it is moving north.

2. A ship sails N. E. by N. a distance of 578 miles. Find its departure and difference in latitude.

3. A ship sails on a course of 73° until its departure is 315 miles. Find the actual distance sailed. Find also its difference in latitude.

4. A ship sails from latitude 47° 15′ N. 670 miles on a course N. W. by N. Find the latitude arrived at.

5. A ship sails from latitude 30° 24′ N. and after 25 hours reaches latitude 35° 26′ N. Its course was N. N. W. Find the average speed of the ship.

6. A vessel sails from lat. 24° 30′ N., long. 30° 15′ W., a distance of 692 miles on a course of 32° 20′. Find the latitude and longitude of its new position.

7. A vessel sails from lat. 10° 30′ S., long. 167° 20′ W., a distance of 692 miles on a course of 152° 30′. Find the latitude and longitude of its new position.

CHAPTER III

SIMPLE TRIGONOMETRIC RELATIONS

23. Other Trigonometric Functions. The reciprocals of the sine, the cosine, and the tangent of any angle are called, respectively, the cosecant, the secant, and the cotangent of that angle. Thus,

$$\text{cosecant}\ \theta = \frac{\text{distance of } P}{\text{ordinate of } P} = \frac{r}{y} \quad (\text{provided } y \neq 0).$$

$$\text{secant}\ \theta = \frac{\text{distance of } P}{\text{abscissa of } P} = \frac{r}{x} \quad (\text{provided } x \neq 0).$$

$$\text{cotangent}\ \theta = \frac{\text{abscissa of } P}{\text{ordinate of } P} = \frac{x}{y} \quad (\text{provided } y \neq 0).$$

These functions are written $\csc\theta$, $\sec\theta$, $\text{ctn}\,\theta$. From the definitions follow directly the relations

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \text{ctn}\,\theta = \frac{1}{\tan\theta};$$

or

$$\csc\theta\cdot\sin\theta = 1, \quad \sec\theta\cdot\cos\theta = 1, \quad \text{ctn}\,\theta\cdot\tan\theta = 1.$$

To the above functions may be added versed sine (written versin), the coversed sine (written coversin), and the external secant (written exsec), which are defined by the equations $\text{versin}\ \theta = 1 - \cos\theta$, $\text{coversin}\ \theta = 1 - \sin\theta$, and $\text{exsec}\ \theta = \sec\theta - 1$. Of importance in navigation and serviceable in other applications (see § 38) is the haversine (written hav) which is defined to be equal to one half the versed sine; *i.e.*

$$\text{hav}\,\theta = \tfrac{1}{2}(1 - \cos\theta).$$

24. The Representation of the Functions by Lines. Consider an angle θ in each quadrant and about the origin draw

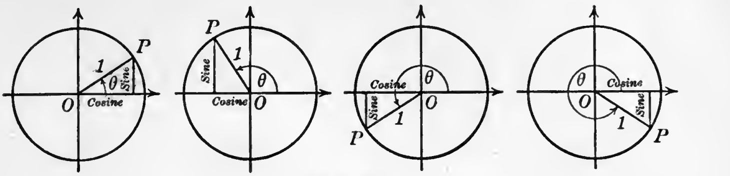


FIG. 24

a circle of unit radius. Let $P(x, y)$ be the point where the circle meets the terminal side of θ. Then

$$\sin\theta = \frac{y}{1} = y,\quad \cos\theta = \frac{x}{1} = x,$$

i.e. the sine is represented by the ordinate of P and the cosine by the abscissa. Hence the sine and cosine have respectively the same signs as the ordinate and abscissa of P.

If we draw a tangent to the circle at the point A where the

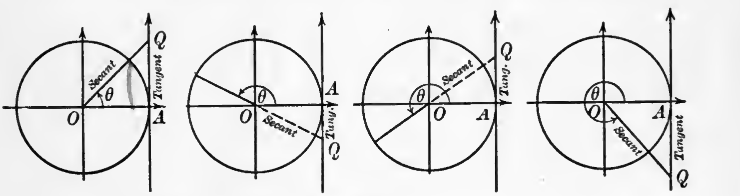


FIG. 25

circle meets the x-axis and let the terminal line of θ meet this tangent in Q, we have

$$\tan\theta = \frac{AQ}{1} = AQ,\quad \sec\theta = \frac{OQ}{1} = OQ.$$

Note that when $\theta = 90°$, $270°$, and in general $90 + n \cdot 360°$, $270° + n \cdot 360°$, where n is any integer, there is no length AQ cut off on the tangent line and hence these angles have no tangents.

If we draw a line tangent to the circle at the point B where the circle cuts the y-axis and let the terminal line of θ cut this tangent in R, we have

$$\operatorname{ctn}\theta = \frac{BR}{1} = BR, \text{ and } \csc\theta = \frac{OR}{1} = OR.$$

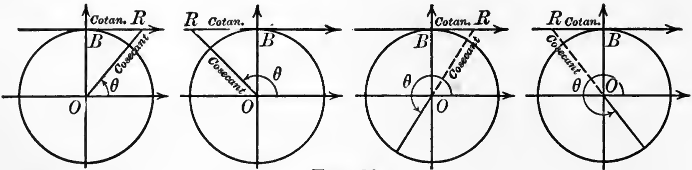


FIG. 26

EXERCISES

1. From Fig. 24 prove $\sin^2\theta + \cos^2\theta = 1$.
2. From Fig. 25 prove $1 + \tan^2\theta = \sec^2\theta$.
3. From Fig. 26 prove $1 + \operatorname{ctn}^2\theta = \csc^2\theta$.

25. Relations among the Trigonometric Functions. As one might imagine, the six trigonometric functions sine, cosine, tangent, cosecant, secant, cotangent are connected by certain relations. We shall now find some of these relations.

From Fig. 9 (§ 10) it is seen that for all cases we have

(1) $$x^2 + y^2 = r^2.$$

If we divide both sides of (1) by r^2, we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad (\text{by hypothesis } r \neq 0);$$

or

$$\mathbf{\sin^2\theta + \cos^2\theta = 1.}$$

Dividing both sides of (1) by x^2, we have

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2} \quad (\text{if } x \neq 0).$$

Therefore,

$$\mathbf{1 + \tan^2\theta = \sec^2\theta.}$$

Similarly dividing both sides of (1) by y^2 gives

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2} \quad (\text{if } y \neq 0);$$

or

$$\mathbf{\operatorname{ctn}^2\theta + 1 = \csc^2\theta.}$$

Moreover, we have

$$\tan \theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin \theta}{\cos \theta},$$

and, similarly,

$$\operatorname{ctn} \theta = \frac{\cos \theta}{\sin \theta}.$$

26. Identities. By means of the relations just proved any expression containing trigonometric functions may be put into a number of different forms. It is often of the greatest importance to notice that two expressions, although of a different form, are nevertheless identical in value. (How was an "identity" defined in algebra?)

The truth of an identity is usually established by reducing both sides, either to the same expression, or to two expressions which we know to be identical. The following examples will illustrate the methods used.

EXAMPLE 1. Prove the relation $\sec^2 \theta + \csc^2 \theta = \sec^2 \theta \csc^2 \theta$.

We may write the given equation in the form

$$\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} = \sec^2 \theta \csc^2 \theta,$$

or

$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta} = \sec^2 \theta \csc^2 \theta,$$

or

$$\frac{1}{\cos^2 \theta \sin^2 \theta} = \sec^2 \theta \csc^2 \theta,$$

which reduces to

$$\sec^2 \theta \csc^2 \theta = \sec^2 \theta \csc^2 \theta.$$

Since this is an identity, it follows, by retracing the steps, that the given equality is identically true.

Both members of the given equality are undefined for the angles 0°, 90°, 180°, 270°, 360°, or any multiples of these angles.

EXAMPLE 2. Prove the identity $1 + \sin \theta = \dfrac{\cos^2 \theta}{1 - \sin \theta}$.

Since $\cos^2 \theta = 1 - \sin^2 \theta$, we may write the given equation in the form $1 + \sin \theta = \dfrac{1 - \sin^2 \theta}{1 - \sin \theta}$ or $1 + \sin \theta = 1 + \sin \theta$.

As in Example 1, this shows that the given equality is identically true. The right-hand member has no meaning when $\sin\theta = 1$, while the left-hand member is defined for all angles. We have, therefore, proved that the two members are equal except for the angle $90°$ or $(4n+1)90°$, where n is any integer.

The formulas of § 25 may be used to solve examples of the type given in § 16.

EXAMPLE 3. Given that $\sin\theta = \frac{5}{13}$ and that $\tan\theta$ is negative, find the values of the other trigonometric functions.

Since $\sin^2\theta + \cos^2\theta = 1$, it follows that $\cos\theta = \pm\frac{12}{13}$, but since $\tan\theta$ is negative, θ lies in the second quadrant and $\cos\theta$ must be $-\frac{12}{13}$. Moreover, the relation $\tan\theta = \sin\theta/\cos\theta$ gives $\tan\theta = -\frac{5}{12}$. The reciprocals of these functions give $\sec\theta = -\frac{13}{12}$, $\csc\theta = \frac{13}{5}$, $\text{ctn}\,\theta = -\frac{12}{5}$.

EXERCISES

1. Define secant of an angle; cosecant; cotangent.

2. Are there any angles for which the secant is undefined? If so, what are the angles? Answer the same question for cosecant and cotangent.

3. Define versed sine; coversed sine; haversine.

4. Complete the following formulas:

$$\sin^2\theta + \cos^2\theta = ? \quad 1 + \tan^2\theta = ? \quad 1 + \text{ctn}^2\,\theta = ? \quad \tan\theta = ?$$

Do these formulas hold for all angles?

5. In what quadrants is the secant positive? negative? the cosecant positive? negative? cotangent positive? negative?

6. Is there an angle whose tangent is positive and whose cotangent is negative?

7. In what quadrant is an angle situated if we know that

(*a*) its sine is positive and its cotangent is negative?

(*b*) its tangent is negative and its secant is positive?

(*c*) its cotangent is positive and its cosecant is negative?

8. Express $\sin^2\theta + \cos\theta$ so that it shall contain no trigonometric function except $\cos\theta$.

9. Transform $(1 + \text{ctn}^2\,\theta)\csc\theta$ so that it shall contain only $\sin\theta$.

10. Which of the trigonometric functions are never less than one in absolute value?

11. For what angles is the following equation true: $\tan\theta = \text{ctn}\,\theta$?

12. How many degrees are there in θ when $\text{ctn}\,\theta = 1$? $\text{ctn}\,\theta = -1$? $\sec\theta = \sqrt{2}$? $\csc\theta = \sqrt{2}$?

13. Determine from a figure the values of the secant, cosecant, and cotangent of 30°, 150°, 210°, 330°.

14. Determine from a figure the values of the secant, cosecant, and cotangent of 45°, 135°, 225°, 315°.

15. Determine from a figure the values of the sine, cosine, tangent, secant, cosecant, and cotangent of 60°, 120°, 240°, 300°.

16. Find θ from the following equations.

(*a*) $\sin\theta = \frac{1}{2}$.
(*b*) $\sin\theta = -\frac{1}{2}$.
(*c*) $\cos\theta = \frac{1}{2}$.
(*d*) $\cos\theta = -\frac{1}{2}$.
(*e*) $\sec\theta = 2$.
(*f*) $\sec\theta = -2$.
(*g*) $\csc\theta = 2$.
(*h*) $\csc\theta = -2$.
(*i*) $\tan\theta = -1$.
(*j*) $\operatorname{ctn}\theta = -1$.
(*k*) $\tan\theta = 1$.
(*l*) $\operatorname{ctn}\theta = 1$.
(*m*) $\tan^2\theta = 3$.
(*n*) $\sin\theta = 0$.
(*o*) $\cos\theta = 0$.
(*p*) $\tan\theta = 0$.

Prove the following identities and state for each the exceptional values of the variables, if any, for which one or both members are undefined:

17. $\cos\theta\tan\theta = \sin\theta$.

18. $\sin\theta\operatorname{ctn}\theta = \cos\theta$.

19. $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}$.

20. $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1$.

21. $(1-\sin^2\theta)\csc^2\theta = \operatorname{ctn}^2\theta$.

22. $\tan\theta + \operatorname{ctn}\theta = \sec\theta\csc\theta$.

23. $[x\sin\theta + y\cos\theta]^2 + [x\cos\theta - y\sin\theta]^2 = x^2 + y^2$.

24. $\dfrac{\csc\theta}{\tan\theta + \operatorname{ctn}\theta} = \cos\theta$.

25. $1 - \operatorname{ctn}^4\theta = 2\csc^2\theta - \csc^4\theta$.

26. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.

27. $2(1+\sin\theta)(1+\cos\theta) = (1+\sin\theta+\cos\theta)^2$.

28. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$.

29. $\dfrac{\csc\theta}{\csc\theta - 1} + \dfrac{\csc\theta}{\csc\theta + 1} = 2\sec^2\theta$.

30. $\dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\operatorname{ctn}\theta - 1}{\operatorname{ctn}\theta + 1}$.

27. The Trigonometric Functions of $90° - \theta$. Figure 27 represents angles θ and $90° - \theta$, when θ is in each of the four quadrants. Let OP be the terminal line of θ and OP' the terminal line of $90° - \theta$. Take $OP' = OP$ and let (x, y) be

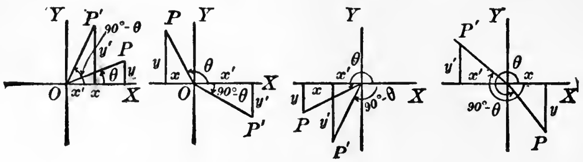


FIG. 27

the coördinates of P and (x', y') the coördinates of P'. Then in all four figures we have

$$x' = y, \quad y' = x, \quad r' = r.$$

Hence

$$\sin(90° - \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° - \theta) = \frac{x'}{r} = \frac{y}{r} = \sin\theta,$$

$$\tan(90° - \theta) = \frac{y'}{x'} = \frac{x}{y} = \operatorname{ctn}\theta.$$

Also,

$$\csc(90° - \theta) = \sec\theta,$$
$$\sec(90° - \theta) = \csc\theta,$$
$$\operatorname{ctn}(90° - \theta) = \tan\theta.$$

DEFINITION. The sine and cosine, the tangent and cotangent, the secant and cosecant, are called *co-functions* of each other.

The above results may be stated as follows: *Any function of an angle is equal to the corresponding co-function of the complementary angle.**

28. The Trigonometric Functions of 180° − θ. By drawing figures as in § 27, the following relations may be proved:

$\sin(180° - \theta) = \sin\theta$, $\csc(180° - \theta) = \csc\theta$,
$\cos(180° - \theta) = -\cos\theta$, $\sec(180° - \theta) = -\sec\theta$,
$\tan(180° - \theta) = -\tan\theta$, $\operatorname{ctn}(180° - \theta) = -\operatorname{ctn}\theta$.

The proof is left as an exercise.

* Two angles are said to be *complementary* if their sum is 90°, regardless of the size of the angles.

29. The result of § 27 shows why it is possible to arrange the tables of the trigonometric functions with angles from 0° to 45° at the top of the pages and angles from 45° to 90° at the bottom of the pages. For example, since $\sin(90° - \theta) = \cos\theta$, the entry for $\cos\theta$ will serve equally well for $\sin(90° - \theta)$. As particular instances we may note $\sin 67° = \cos 23°$, $\tan 67° = \text{ctn}\, 23°$, $\cos 67° = \sin 23°$. Verify these from the table.

The result of § 28 enables us to find the values of the functions of an obtuse angle from tables that give the values only for acute angles. It will be noted that § 28 says that *any function of an obtuse angle is in absolute value equal to the same function of its supplementary angle but may differ from it in sign.*

Thus to find tan 137° we know that it is in absolute value the same as $\tan(180° - 137°) = \tan 43° = 0.9325$. But tan 137° is negative. Hence

$$\tan 137° = -0.9325.$$

Similarly,

$$\sin 137° = 0.6820.$$
$$\cos 137° = -0.7314.$$

EXERCISES

Find the values of the following:

tan 146°, sin 136°, cos 173°, tan 100°, cos 96°, sin 138°,
tan 98°, sin 145°, cos 168°, cos 138°, tan 173°, cos 157°.

$F(90° - \theta) = \text{cof}(\theta)$

$F(90° + \theta) = \pm \text{cof}(\theta)$

$F(180° - \theta) = \pm F(\theta)$

CHAPTER IV

OBLIQUE TRIANGLES

30. Law of Sines. Consider any triangle ABC with the altitude CD drawn from the vertex C (Fig. 28).

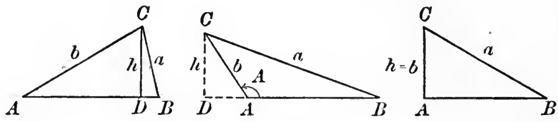


FIG. 28

In all cases we have $\sin A = \frac{h}{b}, \ \sin B = \frac{h}{a}.$ (1)

Therefore, dividing, we obtain

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B}. \tag{2}$$

If the perpendicular were dropped from B, the same argument would give $a/\sin A = c/\sin C$. Hence, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This law is known as the ***law of sines*** and may be stated as follows: *Any two sides of a triangle are proportional to the sines of the angles opposite these sides.*

31. Law of Cosines. Consider any triangle ABC with the altitude CD drawn from the vertex C (Fig. 29).

In Fig. 29 *a*

$$AD = b \cos A; \ CD = b \sin A; \ DB = c - b \cos A.$$

In Fig. 29 *b*

$$AD = -b \cos A; \ CD = b \sin A; \ DB = c - b \cos A.$$

In both figures

$$a^2 = DB^2 + CD^2.$$

Therefore

$$a^2 = c^2 - 2\,bc\cos A + b^2\cos^2 A + b^2\sin^2 A$$
$$= c^2 - 2\,bc\cos A + (\cos^2 A + \sin^2 A)b^2,$$

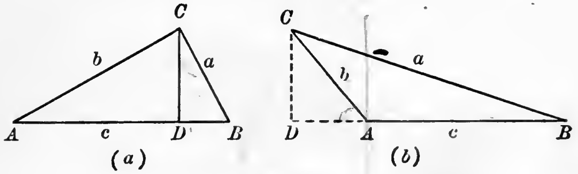


FIG. 29

whence

$$\mathbf{a^2 = b^2 + c^2 - 2\,bc\cos A.}$$

The result holds also when A is a right angle. Why?

Similarly it may be shown that

$$b^2 = c^2 + a^2 - 2\,ca\cos B,$$
$$c^2 = a^2 + b^2 - 2\,ab\cos C.$$

Any one of these similar results is called the ***law of cosines.*** It may be stated as follows:

*The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of these two sides times the cosine of their included angle.**

32. Solution of Triangles. To solve a triangle is to find the parts not given, when certain parts are given. From geometry we know that a triangle is in general determined when three parts of the triangle, one of which is a side, are given.† Right triangles have already been solved (§ 15), and we shall now make use of the laws of sines and cosines to solve oblique triangles. The methods employed will be illustrated by some examples. It will be found advantageous to construct the triangle to scale, for by so doing one can often detect errors which may have been made.

* Of what three theorems in elementary geometry is this the equivalent?

† When two sides and an angle opposite one of them are given, the triangle is not always determined. Why?

33. Illustrative Examples.

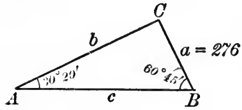


FIG. 30

EXAMPLE 1. Solve the triangle ABC, given $A = 30°\ 20'$, $B = 60°\ 45'$, $a = 276$.

SOLUTION :

$$C = 180° - (A + B) = 180° - 91°\ 5' = 88°\ 55';$$

$$b = \frac{a \sin B}{\sin A} = \frac{276 \sin 60°\ 45'}{\sin 30°\ 20'} = \frac{(276)(0.8725)}{0.5050} = 476.9 :$$

also

$$c = \frac{a \sin C}{\sin A} = \frac{276 \sin 88°\ 55'}{\sin 30°\ 20'} = \frac{(276)(0.9998)}{0.5050} = 546.4.$$

CHECK : It is left as an exercise to show that for these values we have

$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

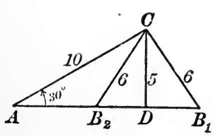


FIG. 31

EXAMPLE 2. Solve the triangle ABC, given $A = 30°$, $b = 10$, $a = 6$.

Constructing the triangle ABC, we see that two triangles AB_1C and AB_2C answer the description since $b > a >$ altitude CD.

SOLUTION : Now

$$\frac{\sin B_1}{\sin A} = \frac{b}{a}, \text{ or } \sin B_1 = \frac{b \sin A}{a} = 0.833,$$

whence $\qquad B_1 = 56°.5.$

But

$$B_2 = 180° - B_1 = 180° - 56°.5 = 123°.5,$$

and

$$C_1 = 180° - (A + B_1) = 180° - 86°.5 = 93°.5,$$
$$C_2 = 180° - (A + B_2) = 180° - 153°.5 = 26°.5.$$

Now

$$\frac{c_2}{a} = \frac{\sin C_2}{\sin A}, \quad \text{or} \quad c_2 = \frac{a \sin C_2}{\sin A} = \frac{(6)(0.446)}{0.500} = 5.35.$$

Also

$$\frac{c_1}{a} = \frac{\sin C_1}{\sin A}; \quad \text{or} \quad c_1 = \frac{a \sin C_2}{\sin A} = \frac{(6)(0.998)}{0.500} = 11.98.$$

CHECK :

$$c_1^2 = a^2 + b^2 - 2\,ab \cos C_1.$$
$$143.5 = 36 + 100 + (2)(6)(10)(0.061) = 143.3.$$
$$c_2^2 = a^2 + b^2 - 2\,ab \cos C_2.$$
$$28.62 = 36 + 100 - (2)(6)(10)(0.895) = 28.60.$$

EXAMPLE 3. Solve the triangle ABC, given $a = 10$, $b = 6$, $C = 40°$.

SOLUTION: $c^2 = a^2 + b^2 - 2ab\cos C$

$= 100 + 36 - (120)(0.766) = 44.08.$

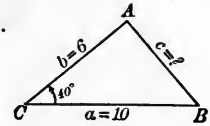


FIG. 32

Therefore $c = 6.64$. Now

$$\sin A = \frac{a \sin C}{c} = \frac{(10)(0.643)}{6.64} = 0.968,$$

i.e. $A = 104°.5$. Likewise,

$$\sin B = \frac{b \sin C}{c} = \frac{(6)(0.643)}{6.64} = 0.581,$$

i.e. $B = 35°.5$.

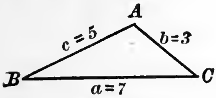


FIG. 33

CHECK: $A + B + C = 180°.0$.

EXAMPLE 4. Solve the triangle ABC when $a = 7$, $b = 3$, $c = 5$.

From the law of cosines,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{1}{2} = -0.500,$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{13}{14} = 0.928,$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{11}{14} = 0.786.$$

Therefore

$$A = 120°,\ B = 21°.8,\ C = 38°.2.$$

CHECK: $A + B + C = 180°.0$

EXERCISES

1. Solve the triangle ABC, given

(a) $A = 30°$, $B = 70°$, $a = 100$;
(b) $A = 40°$, $B = 70°$, $c = 110$;
(c) $A = 45°.5$, $C = 68°.5$, $b = 40$;
(d) $B = 60°.5$, $C = 44° 20'$, $c = 20$;
(e) $a = 30$, $b = 54$, $C = 50°$;
(f) $b = 8$, $a = 10$, $C = 60°$;
(g) $a = 10$, $b = 12$, $c = 14$;
(h) $a = 21$, $b = 24$, $c = 28$.

2. Determine the number of solutions of the triangle ABC when

(a) $A = 30°$, $b = 100$, $a = 70$;
(b) $A = 30°$, $b = 100$, $a = 100$;
(c) $A = 30°$, $b = 100$, $a = 50$;
(d) $A = 30°$, $b = 100$, $a = 40$;
(e) $A = 30°$, $b = 100$, $a = 120$;
(f) $A = 106°$, $b = 120$, $a = 16$;
(g) $A = 90°$, $b = 15$, $a = 14$.

3. Solve the triangle ABC when

(a) $A = 37° 20'$, $a = 20$, $b = 26$; (c) $A = 30°$, $a = 22$, $b = 34$.
(b) $A = 37° 20'$, $a = 40$, $b = 26$;

4. In order to find the distance from a point A to a point B, a line AC and the angles CAB and ACB were measured and found to be 300 yd., $60° 30'$, $56° 10'$ respectively. Find the distance AB.

5. In a parallelogram one side is 40 and one diagonal 90. The angle between the diagonals (opposite the side 40) is 25°. Find the length of the other diagonal and the other side. How many solutions?

6. Two observers 4 miles apart, facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 60° and 40° respectively. Find the distance from the balloon to each observer and the height of the balloon.

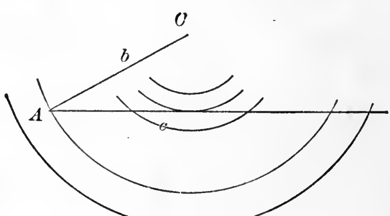
7. Two stakes A and B are on opposite sides of a stream; a third stake C is set 100 feet from A, and the angles ACB and CAB are observed to be 40° and 110°, respectively. How far is it from A to B?

8. The angle between the directions of two forces is 60°. One force is 10 pounds and the resultant of the two forces is 15 pounds. Find the other force.*

9. Resolve a force of 90 pounds into two equal components whose directions make an angle of 60° with each other.

10. An object B is wholly inaccessible and invisible from a certain point A. However, two points C and D on a line with A may be found such that from these points B is visible. If it is found that $CD = 300$ feet, $AC = 120$ feet, angle $DCB = 70°$, angle $CDB = 50°$, find the length AB.

11. Given a, b, A, in the triangle ABC. Show that the number of possible solutions are as follows:



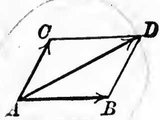
$A < 90°$

$$\begin{cases} a < b \sin A & \text{no solution,} \\ b \sin A < a < b & \text{two solutions,} \\ \left.\begin{matrix} a \geqq b \\ a = b \sin A \end{matrix}\right\} & \text{one solution.} \end{cases}$$

$A \geqq 90°$

$$\begin{cases} a \leqq b & \text{no solution,} \\ a > b & \text{one solution.} \end{cases}$$

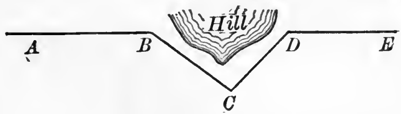
12. The diagonals of a parallelogram are 14 and 16 and form an angle of 50°. Find the length of the sides.



* It is shown in physics that if the line segments AB and AC represent in magnitude and direction two forces acting at a point A, then the diagonal AD of the parallelogram $ABCD$ represents both in magnitude and direction the resultant of the two given forces.

13. Resolve a force of magnitude 150 into two components of 100 and 80 and find the angle between these components.

14. It is sometimes desirable in surveying to extend a line such as AB

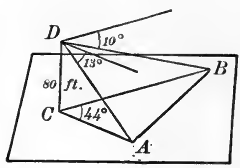


in the adjoining figure. Show that this can be done by means of the broken line $ABCDE$. What measurements are necessary?

15. Three circles of radii 2, 6, 5 are mutually tangent. Find the angles between their lines of centers.

16. In order to find the distance between two objects A and B on opposite sides of a house, a station C was chosen, and the distances $CA = 500$ ft., $CB = 200$ ft., together with the angle $ACB = 65° 30'$, were measured. Find the distance from A to B.

17. The sides of a field are 10, 8, and 12 rods respectively. Find the angle opposite the longer side.



18. From a tower 80 feet high, two objects, A and B, in the plane of the base are found to have angles of depression of 13° and 10° respectively; the horizontal angle subtended by A and B at the foot C of the tower is 44°. Find the distance from A to B.

34. Areas of Oblique Triangles.

1. *When two sides and the included angle are given.*

Denoting the area by S, we have from geometry

$$S = \tfrac{1}{2} ch,$$

but $h = b \sin A$; therefore

$$S = \tfrac{1}{2} cb \sin A. \quad (1)$$

Likewise,

$$S = \tfrac{1}{2} ab \sin C \text{ and } S = \tfrac{1}{2} ac \sin B.$$

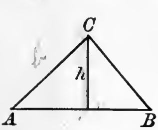


FIG. 34

2. *When a side and two adjacent angles are given.*

Suppose the side a and the adjacent angles B and C to be given. We have just seen that $S = \tfrac{1}{2} ac \sin B$. But from the law of sines we have

$$c = \frac{a \sin C}{\sin A}.$$

Therefore

$$S = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \sin A}.$$

But $\sin A = \sin[180° - (B + C)] = \sin(B + C)$. Therefore

$$S = \frac{a^2 \sin B \sin C}{2 \sin(B + C)}.$$

3. *When the three sides are given.*

We have seen that $S = \frac{1}{2} bc \sin A$. Squaring both sides of this formula and transforming, we have

$$S^2 = \frac{b^2c^2}{4} \sin^2 A = \frac{b^2c^2}{4}(1 - \cos^2 A)$$

$$= \frac{bc}{2}(1 + \cos A) \cdot \frac{bc}{2}(1 - \cos A);$$

whence, by the law of cosines,

$$S^2 = \frac{bc}{2}\left(1 + \frac{b^2 + c^2 - a^2}{2\,bc}\right) \cdot \frac{bc}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2\,bc}\right)$$

$$= \frac{2\,bc + b^2 + c^2 - a^2}{4} \cdot \frac{2\,bc - b^2 - c^2 + a^2}{4}$$

$$= \frac{b + c + a}{2} \cdot \frac{b + c - a}{2} \cdot \frac{a - b + c}{2} \cdot \frac{a + b - c}{2},$$

which may be written in the form

$$S^2 = s(s - a)(s - b)(s - c),$$

where $2\,s = a + b + c$. Therefore,

(2) $$S = \sqrt{s(s - a)(s - b)(s - c)}.$$

35. The Radius of the Inscribed Circle. If r is the radius of the inscribed circle, we have from elementary geometry, since s is half the perimeter of the triangle, $S = rs$; equating this value of S to that found in equation (2) of the last article and then solving for r, we get,

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

EXERCISES

Find the area of the triangle ABC, given

1. $a = 25$, $b = 31.4$, $C = 80° 25'$.
2. $b = 24$, $c = 34.3$, $A = 60° 25'$.
3. $a = 37$, $b = 13$, $C = 40°$.
4. $a = 10$, $b = 7$, $C = 60°$.
5. $a = 10$, $b = 12$, $C = 60°$.
6. $a = 10$, $b = 12$, $C = 8°$.

7. Find the area of a parallelogram in terms of two adjacent sides and the included angle.

8. The base of an isosceles triangle is 20 ft. and the area is $100/\sqrt{3}$ sq. ft. Find the angles of the triangle. *Ans.* 30°, 30°, 120°.

9. Find the radius of the inscribed circle of the triangle whose sides are 12, 10, 8.

10. How many acres are there in a triangular field having one of its sides 50 rods in length and the two adjacent angles, respectively, 70° and 60°?

36. The Law of Tangents. For the work in the next chapter the formulas in this and the next article will be needed.

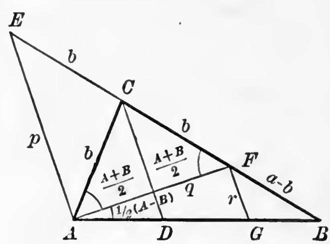


FIG. 35

Let CD be the bisector of the angle C of the $\triangle ABC$. Through A draw a line $\parallel DC$, meeting BC produced in E. Then $CE = b$. Why? From A draw a line $q \perp DC$ meeting CB in F. At F draw a line $r \perp AF$ meeting AB in G. Let $AE = p$.

Now $\triangle ACF$ is isosceles. Why? The angle $ACE = \angle A + \angle B$ and the bisector of $\angle ACE$ is $\perp CD$. Hence $\angle CAF = \angle CFA = \frac{1}{2}\angle(A+B)$. Moreover $\angle BAF = \angle A - \frac{1}{2}\angle(A+B) = \frac{1}{2}\angle(A-B)$.

Now $$\tan\frac{A+B}{2} = \frac{p}{q} \text{ and } \tan\frac{A-B}{2} = \frac{r}{q}.$$

$$\therefore \quad \frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} = \frac{p}{r}.$$

But $$\frac{p}{r} = \frac{BE}{BF} = \frac{a+b}{a-b}. \quad \text{Why?}$$

Hence $$\frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}} = \frac{a+b}{a-b}.$$

37. Angles of a Triangle in Terms of the Sides. Construct the inscribed circle of the triangle and denote its radius by r. If the perimeter $a + b + c = 2s$, then (Fig. 36)

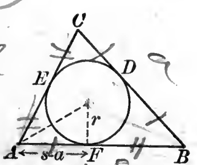


FIG. 36

$$AE = AF = s - a.$$
$$BD = BF = s - b.$$
$$CD = CE = s - c.$$

Then $\tan \frac{1}{2} A = \frac{r}{s-a}$, $\tan \frac{1}{2} B = \frac{r}{s-b}$, $\tan \frac{1}{2} C = \frac{r}{s-c}$,

where, from § 35,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

38. Solution of Triangles by Means of the Haversine. The haversine may be used advantageously in the solution of triangles, (1) when two sides and the included angle are given; (2) when the three sides are given. The law of cosines gives

$$2 \operatorname{hav} A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{a^2 - (b-c)^2}{2bc}$$

or $$4bc \operatorname{hav} A = a^2 - (b-c)^2.$$

1. If b, c and A are given we may find a from the formula

(1) $$\boldsymbol{a^2 = (b-c)^2 + 4bc \operatorname{hav} A}.$$

Similar formulas give b^2 or c^2 in terms of a, c, B and a, b, C respectively.

2. If a, b, c are given, we may find A from the formula

(2) $$\operatorname{hav} A = \frac{a^2 - (b-c)^2}{4bc} = \frac{(s-b)(s-c)}{bc}.$$

Similar formulas will give B and C.

Example 1. Given $A = 94° \, 23'.4$, $b = 55.12$, $c = 39.90$. To find a. By formula (1) above:

$$\begin{aligned} b &= 55.12 \\ c &= 39.90 \\ (b-c) &= 15.22 \\ (b-c)^2 &= 231.6 \\ 4\,bc \text{ hav } A &= 4736 \\ a^2 &= 4968 \\ a &= 70.49 \end{aligned} \qquad \begin{aligned} bc &= 2199 \\ \text{hav } 94° \, 23'.4 &= 0.0446 \\ bc \text{ hav } A &= 1184 \\ 4\,bc \text{ hav } A &= 4736 \end{aligned}$$

Example 2. Given $a = 4.51$, $b = 6.13$, $c = 8.16$. Find A, B, C.

$$\begin{aligned} a^2 &= 20.34 \\ (b-c)^2 &= 4.12 \\ a^2 - (b-c)^2 &= 16.22 \\ bc &= 50.02 \\ 4\,bc &= 200.1 \end{aligned} \qquad \text{hav } A = \frac{16.22}{200.1} = 0.0811 \qquad A = 33° \, 05$$

$$\begin{aligned} b^2 &= 37.58 \\ (c-a)^2 &= 13.32 \\ b^2 - (c-a)^2 &= 24.26 \\ ac &= 36.80 \\ 4\,ac &= 147.21 \end{aligned} \qquad \text{hav } B = \frac{24.26}{147.21} = 0.1648 \qquad B = 47° \, 54$$

$$\begin{aligned} c^2 &= 66.59 \\ (b-a)^2 &= 2.62 \\ c^2 - (b-a)^2 &= 63.97 \\ ab &= 27.646 \\ 4\,ab &= 110.58 \end{aligned} \qquad \text{hav } C = \frac{63.97}{110.58} = 0.5785 \qquad C = 99° \, 02'$$

Check: $180° \, 01'$

EXERCISES

Solve the following triangles:

1. $a = 62.1$, $b = 32.7$, $c = 47.2$.
2. $A = 37° \, 20'$, $b = 2.4$, $c = 4.7$.
3. $B = 121° \, 32'$, $a = 27.9$, $c = 35.8$.
4. $a = 3.2$, $b = 5.7$, $c = 6.5$.
5. $C = 72° \, 21'.4$, $a = 314.1$, $b = 427.3$.
6. $a = 346.1$, $b = 425.8$, $c = 562.3$.

CHAPTER V

LOGARITHMS

39. The Invention of Logarithms. The extensive numerical computations required in business, in science, and in engineering were greatly simplified by the invention of **logarithms** by John Napier, Baron of Merchiston (1550–1617). By means of logarithms we are able to replace multiplication and division by addition and subtraction, processes which we all realize are more expeditious than the first two.

If we consider the successive integral powers of 2

(1)

Exponent x .	1	2	3	4	5	6	7
Result 2^x . .	2	4	8	16	32	64	128

Exponent x .	8	9	10	11	12	etc.	A. P.
Result 2^x . .	256	512	1024	2048	4096	etc.	G. P.

we see that the results form a geometric progression (G. P.) and the exponents an arithmetic progression (A. P.). We know from elementary algebra that

$$x^m \cdot x^n = x^{m+n},$$

and

$$\frac{x^m}{x^n} = x^{m-n}.$$

Hence if we wish to multiply two numbers in our G. P. *e.g.* 4×8, we merely have to add the corresponding exponents 2 and 3 and under the sum 5 find the desired product 32. Similarly, if we wish to divide *e.g.* 4096 by 128, we merely have to subtract the exponent corresponding to 128, from that corresponding to 4096 and under their difference 5 we find the desired quotient 32.

To make the above plan at all useful it is evident that our table must be expanded so as to contain more numbers. First we can expand our table so that it will contain numbers less than 2, by subtracting 1 successively from the numbers in the A. P. and by dividing successively by 2 the numbers in the G. P.

(2)

− 5	− 4	− 3	− 2	− 1	0	1	2	3	4	5	6	7
0.03125	0.0625	0.125	0.25	0.5	1	2	4	8	16	32	64	128

In the second place we may find new numbers by inserting arithmetic means and geometric means. Thus, if we take the following portion of the preceding table

− 2	− 1	0	1	2	3	4
$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

and insert between every two successive numbers of the upper line their arithmetic, and between every two successive numbers of the lower line their geometric mean, we obtain the table

(3)

− 2	$-\frac{3}{2}$	− 1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{1}{4}$	$\frac{1}{4}\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	1	$\sqrt{2}$	2	$2\sqrt{2}$	4	$4\sqrt{2}$	8	$8\sqrt{2}$	16

If the radicals are expressed approximately as decimals, this table takes the form

− 2.0	− 1.5	− 1.0	− 0.5	0	0.5	1.0	1.5	2	2.5	3	3.5	4
0.25	0.35	0.50	0.72	1.00	1.41	2.00	2.83	4.00	5.66	8.00	11.31	16

By continuing this process we can make any number appear in the G. P. to as high a degree of approximation as we desire. To prepare an extensive table, which gives values at small intervals, is quite laborious. However, it has been done, and we have printed tables so complete that actual multiplication of any two numbers can be replaced by addition of two other numbers. We shall soon learn how to use such tables.

40. Definition of the Logarithm. The **logarithm** of a number N to a base $b\,(b > 0,\ \neq 1)$ is the exponent x of the power to which the base b must be raised to produce the number N.

That is, if

$$b^x = N,$$

then

$$x = \log_b N.$$

These two equations are of the highest importance in all work concerning logarithms. One should keep in mind the fact that if either of them is given, the other may always be inferred.

The numbers forming the A. P. in tables 1, 2, and 3 of § 39 are the logarithms of the corresponding numbers in the G. P., the base being 2. From table 3 we have $2^{\frac{5}{2}} = 4\sqrt{2}$ which says $\log_2 4\sqrt{2} = \frac{5}{2}$.

EXERCISES

1. When 3 is the base what are the logarithms of 9, 27, 3, 1, 81, $\frac{1}{3}$, $\frac{1}{243}$, $27^{\frac{1}{2}}$?

2. Why cannot 1 be used as the base of a system of logarithms?

3. When 10 is the base what are the logarithms of 1, 10, 100, 1000?

4. Find the values of x which will satisfy each of the following equalities:

(*a*) $\log_3 27 = x$.
(*b*) $\log_x 3 = 1$.
(*c*) $\log_x 5 = \frac{1}{2}$.
(*d*) $\log_a a = x$.
(*e*) $\log_a 1 = x$.
(*f*) $\log_3 \frac{1}{81} = x$.
(*g*) $\log_2 x = 6$.
(*h*) $\log_{32} x = \frac{1}{5}$.
(*i*) $\log_{0.001} x = 2$.

5. Find the value of each of the following expressions:

(*a*) $\log_2 16$. (*c*) $\log_6 \frac{1}{216}$. (*e*) $\log_{25} 125$.
(*b*) $\log_{343} 49$. (*d*) $\log_2 \sqrt{16}$. (*f*) $\log_2 \frac{1}{64}$.

41. The Three Fundamental Laws of Logarithms. From the laws of exponents we derive the following fundamental laws.

I. *The logarithm of a product equals the sum of the logarithms of its factors.* Symbolically,

$$\log_b MN = \log_b M + \log_b N.$$

PROOF. Let $\log_b M = x$, then $b^x = M$. Let $\log_b N = y$, then $b^y = N$. Hence we have $MN = b^{x+y}$, or

$$\log_b MN = x + y, \quad \text{i.e.} \quad \log_b MN = \log_b M + \log_b N.$$

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.* Symbolically,

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

PROOF. Let $\log_b M = x$, then $b^x = M$. Let $\log_b N = y$, then $b^y = N$. Hence we have $M/N = b^{x-y}$, or

$$\log_b \frac{M}{N} = x - y, \quad \text{i.e.} \quad \log_b \frac{M}{N} = \log_b M - \log_b N.$$

III. *The logarithm of the pth power of a number equals p times the logarithm of the number.* Symbolically

$$\log_b M^p = p \log_b M.$$

PROOF. Let $\log_b M = x$, then $b^x = M$. Raising both sides to the pth power, we have $b^{px} = M^p$. Therefore

$$\log_b M^p = px = p \log_b M.$$

From law III it follows that the *logarithm of the real positive nth root of a number is one nth of the logarithm of the number.*

EXERCISES

1. Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, find the value of each of the following expressions:

(*a*) $\log_{10} 6$.

[HINT: $\log_{10} 2 \times 3 = \log_{10} 2 + \log_{10} 3$.]

(*b*) $\log_{10} 21.0$.

(*c*) $\log_{10} 20.0$.

(*d*) $\log_{10} 0.03$.

(*e*) $\log_{10} \frac{7}{2}$.

(*f*) $\log_{10} 5$.

[HINT: $\log_{10} 5 = \log_{10} \frac{10}{2}$.]

(*g*) $\log_{10} 150$.

(*h*) $\log_{10} \sqrt{14}$.

(*i*) $\log_{10} 49$.

(*j*) $\log_{10} \sqrt{2^4 . 7^5}$.

2. Given the same three logarithms as in Ex. 1, find the value of each of the following expressions:

(*a*) $\log_{10} \frac{4 \times 5 \times 7}{32 \times 8}$.   (*b*) $\log_{10} \frac{5 \times 3 \times 20}{6 \times 7}$.   (*c*) $\log_{10} \frac{2058}{\sqrt{14}}$.

(*d*) $\log_{10} (2)^{25}$.   (*e*) $\log_{10} (3)^3(5)^6$,   (*f*) $\log_{10}(2^3)(\frac{1}{9})$.

42. Logarithms to the Base 10. Logarithms to the base 10 are known as common or Briggian logarithms. Proceeding as in § 39 we can show that $10^{0.3010} = 2$, *i.e.* $\log_{10} 2 = 0.3010$. Let us multiply both members of the equation $10^{0.3010} = 2$ by 10, 10^2, 10^3, etc. and notice the effect on the logarithm.

$$10^{0.3010} = 2 \qquad \log_{10} 2 = 0.3010$$
$$10^{1.3010} = 20 \qquad \log_{10} 20 = 1.3010$$
$$10^{2.3010} = 200 \qquad \log_{10} 200 = 2.3010.$$

It should be clear from this example that the decimal part of the logarithm (called the ***mantissa***) of a number greater than 1 depends only on the succession of figures composing the number and not on the position of the decimal point, while the integral part (called the ***characteristic***) depends simply on the position of the decimal point. Hence it is only necessary to tabulate the mantissas, for the characteristics can be found by inspection as the following considerations show.

Since

$10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10{,}000$, etc.

we have $\log_{10} 1 = 0$, $\log_{10} 10 = 1$, $\log_{10} 100 = 2$,

$\log_{10} 1000 = 3$, $\log_{10} 10{,}000 = 4$, etc.

It follows that a number with *one* digit ($\neq 0$) at the left of the decimal point has for its logarithm a number equal to $0 +$ a decimal; a number with *two* digits at the left of its decimal point has for its logarithm a number equal to $1 +$ a decimal; a number with *three* digits at the left of the decimal point has for its logarithm a number equal to $2 +$ a decimal, etc. We conclude, therefore, that *the characteristic of the common logarithm of a number greater than* 1 *is one less than the number of digits at the left of the decimal point.*

Thus, $\log_{10} 456.07 = 2.65903$.

The case of a logarithm of a number less than 1 requires special consideration. Taking the numerical example first considered above, if $\log_{10} 2 = 0.30103$, we have $\log_{10} 0.2 = 0.30103 - 1$. Why? This is a negative number, as it should be (since the logarithms of numbers less than 1 are all negative, if the base is greater than 1). But, if we were to carry out this subtraction and write $\log_{10} 0.2 = -0.69897$ (which would be *correct*), it would change the mantissa, which is *inconvenient*. Hence it is customary to write such a logarithm in the form $9.30103 - 10$.

If there are n ciphers immediately following the decimal point in a number less than 1, the characteristic is $-n-1$. *For convenience, if* $n < 10$, *we write this as* $(9 - n) - 10$. *This characteristic is written in two parts. The first part* $9 - n$ *is written at the left of the mantissa and the* -10 *at the right.*

In the sequel, unless the contrary is specifically stated, we shall assume that all logarithms are to the base 10. We may accordingly omit writing the base in the symbol log when there is no danger of confusion. Thus, the equation $\log 2 = 0.30103$ means $\log_{10} 2 = 0.30103$.

To make practical use of logarithms in computation it is necessary to have a conveniently arranged table from which we can find (*a*) the logarithm of a given number and (*b*) the number corresponding to a given logarithm. The general principles governing the use of tables will be explained by the following examples [Tables, pp. 110, 111].

EXAMPLE 1. Find log 42.7.

The characteristic is 1. In the column headed N (p. 110) we find 42 and if we follow this row across to the column headed 7, we read 6304, which is the desired mantissa. Hence $\log 42.7 = 1.6304$.

EXAMPLE 2. Find log 0.03273.

The characteristic is $8 - 10$. The mantissa cannot be found in our table, but we can obtain it by a process called interpolation. We shall assume that to a small change in the number there corresponds a proportional change in the mantissa. Schematically we have

$$\begin{array}{ccccc} & \text{Number} & & \text{Mantissa} & \\ \text{difference} = 10 \left[\begin{array}{l} 3\left[\begin{array}{l} 3270 \\ 3273 \end{array}\right. \\ \quad 3280 \end{array} \right. & \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} & \left. \begin{array}{r} 5145 \\ ? \\ 5159 \end{array} \right] & 14 = \text{difference} \end{array}$$

Our desired mantissa is $5145 + \frac{3}{10} \cdot 14 = 5149$. Hence $\log 0.03273 = 8.5149 - 10$.

EXAMPLE 3. Find x when $\log x = 0.8485$.

We cannot find this mantissa in our table, but we can find 8482 and 8488 which correspond to 7050 and 7060 respectively. Reversing the process of example 2, we have schematically

$$\begin{array}{ccccc} & \text{Number} & & \text{Mantissa} & \\ \text{Difference} = 10 & \left[\begin{array}{l} 7050 \\ ? \\ 7060 \end{array} \right. & \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} & \left. \begin{array}{r} \left.\begin{array}{r} 8482 \\ 8485 \end{array}\right] 3 \\ 8488 \quad \end{array} \right] & 6 = \text{difference} \end{array}$$

Hence the significant figures in our required number are $7050 + \frac{3}{6} \cdot 10 = 7055$. Since the characteristic is 0 the required number is 7.055.

EXERCISES

1. Find the logarithms of the following numbers from the table on pp. 110, 111 : 482, 26.4, 6.857, 9001, 0.5932, 0.08628, 0.00038.

2. Find the numbers corresponding to the following logarithms: 2.7935, 0.3502, 7.9599 – 10, 9.5300 – 10, 3.6598, 1.0958.

43. Use of Logarithms in Computation. The way in which logarithms may be used in computation will be sufficiently explained in the following examples. A few devices often necessary or at least desirable will be introduced. The latter are usually self-explanatory. Reference is made to them here, in order that one may be sure to note them when they arise. The use of logarithms in computation depends, of course, on the fundamental properties derived in § 41.

EXAMPLE 1. Find the value of $73.26 \times 8.914 \times 0.9214$.

We find the logarithms of the factors, add them, and then find the number corresponding to this logarithm. The work may be arranged as follows:

Numbers		Logarithms
73.26	(→)	1.8649
8.914	(→)	0.9501
0.9214	(→)	9.9645 − 10
		12.7795 − 10
Product = 601.9 *Ans.*	(←)	2.7795

EXAMPLE 2. Find the value of $732.6 \div 89.14$.

Numbers		Logarithms
732.6	(→)	2.8649
89.14	(→)	1.9501
Quotient = 8.219 *Ans.*	(←)	0.9148

EXAMPLE 3. Find the value of $89.14 \div 732.6$.

Numbers		Logarithms
89.14	(→)	11.9501 − 10
732.6	(→)	2.8649
Quotient = 0.1217 *Ans.*	(←)	9.0852 − 10

EXAMPLE 4. Find the value of $\frac{763.2 \times 21.63}{986.7}$.

Whenever an example involves several different operations on the logarithms as in this case, it is desirable to make out a *blank form*. When a blank form is used, all logarithms should be looked up first and entered in their proper places. After this has been done, the necessary operations (addition, subtraction, etc.) are performed. Such a procedure saves time and minimizes the chance of error.

FORM

Numbers			Logarithms
763.3	(→)		. . . . . .
21.63	(→)	(+)	. . . . . .
product			. . . . . .
986.7	(→)	(−)	. . . . . .
. . . . . *Ans.*	(←)		. . . . . .

FORM FILLED IN

Numbers			Logarithms
763.2		(→)	2.8826
21.63		(→)	1.3351
product			4.2177
986.7		(→)	2.9942
16.73	*Ans.*	(←)	1.2235

EXAMPLE 5. Find $(1.357)^5$.

Numbers			Logarithms
1.357		(→)	0.1326
$(1.357)^5 = 4.602$	*Ans.*	(←)	0.6630

EXAMPLE 6. Find the cube root of 30.11.

Numbers			Logarithms
30.11		(→)	1.4787
$\sqrt[3]{30.11} = 3.111$	*Ans.*	(←)	0.4929

EXAMPLE 7. Find the cube root of 0.08244.

Numbers			Logarithms
0.08244		(→)	28.9161 – 30
$\sqrt[3]{0.08244} = 0.4352$	*Ans.*	(←)	9.6387 – 10

EXERCISES

Compute the value of each of the following expressions using the table on pp. 110, 111.

1. 34.96×4.65.
2. $518.7 \times 9.02 \times .0472$.
3. $\dfrac{0.5683}{0.3216}$.
4. $\dfrac{5.007 \times 2.483}{6.524 \times 1.110}$.
5. $(34.16 \times .238)^2$.
6. $8.572 \times 1.973 \times (.8723)^2$.
7. $\sqrt[3]{\dfrac{648.8}{(21.4)^2}}$.
8. $\sqrt{\dfrac{1379}{2791}}$.
9. $\sqrt{\dfrac{2.8076 \times 3.184}{(2.012)^3}}$.
10. $\sqrt[3]{\dfrac{2941 \times 17.32}{2173 \times 18.75}}$.
11. $\dfrac{\sqrt[3]{0.00732}}{\sqrt{735}}$.
12. $(20.027)^{\frac{1}{4}}$.
13. 2^{100}.
14. $\sqrt[3]{150^2 - 100^2}$.
15. $(0.02735)^{\frac{1}{5}}$.
16. $\dfrac{\sqrt{3275}}{(2.01)^{\frac{1}{3}}}$.

44. Cologarithms. Since $\frac{M}{N}$ and $M \cdot \frac{1}{N}$ are equivalent, we may in a logarithmic computation, add the logarithm of $\frac{1}{N}$ instead of subtracting $\log N$. The logarithm of $\frac{1}{N}$ is called the cologarithm of N. Therefore

$$\text{colog } N = \log 1/N = \log 1 - \log N = -\log N,$$

since log 1 is zero.

We write cologarithms, like logarithms, with positive mantissas. Therefore the cologarithm is most easily found by subtracting the logarithm from zero, written in the form 10.0000 − 10.

EXAMPLE. Find the colog 27.3.

$$\begin{array}{rl} & 10.0000 - 10 \\ \log 27.3 = & 1.4362 \\ \hline \text{colog } 27.3 = & 8.5638 - 10 \end{array}$$

The cologarithm can be written down immediately by subtracting the last significant figure of the logarithm from 10 and each of the others from 9. If the logarithm is positive the cologarithm is negative and hence − 10 is affixed.

There is no gain in using cologarithms when we have a quotient of two numbers. There is an advantage when either the numerator or denominator contains two or more factors, for we can save an operation of addi tion or subtraction. Let us solve Ex. 4, § 43, using cologarithms.

EXAMPLE. Find the value of $\frac{763.2 \times 21.63}{986.7}$.

Numbers		Log
763.2	→	2.8826
21.63	→	1.3351
986.7	→	(colog) 7.0058 − 10
16.73	←	1.2235

EXERCISES

Compute the value of each of the following expressions, using cologarithms.

1. $\sqrt{\frac{2.80 \times 37.6}{4.96 \times 23.3}}$.

2. $\sqrt{\frac{97.63 \times 876.5}{2876 \times 3.4 \times 2.987}}$.

3. $\dfrac{5}{7 \times 8 \times 9 \times 27.6}$.

4. $\dfrac{3^{12}}{5^{10} \cdot 2^{7}}$.

5. $\dfrac{\sqrt{3275}}{(2.01)^{\frac{1}{3}}(1.76)^{10}}$.

6. $\dfrac{1293 \times 12\ 7 \times 5}{(1 + \frac{17}{273})(760 + 8)}$.

MISCELLANEOUS EXERCISES

1. What objections are there to the use of a negative number as the base of a system of logarithms?

2. Show that $a^{\log_a x} = x$.

3. Write each of the following expressions as a single term:

(*a*) $\log x + \log y - \log z$.
(*b*) $3 \log x - 2 \log y + 3 \log z$.
(*c*) $3 \log a - \log(x + y) - \frac{1}{2} \log(cx + d) + \log \sqrt{w + x}$.

4. Solve for x the following equations:

(*a*) $2 \log_2 x + \log_2 4 = 1$.
(*b*) $\log_3 x - 3 \log_3 2 = 4$.
(*c*) $2 \log_{10} x - 3 \log_{10} 2 = 4$.
(*d*) $3 \log_2 x + 2 \log_2 3 = 1$.

5. How many digits are there in 2^{35}? 3^{142}? $3^{12} \times 2^{8}$?

6. Which is the greater, $(\frac{21}{20})^{100}$ or 100?

7. Find the value of each of the following expressions:

(*a*) $\log_6 35$. (*b*) $\log_3 34$. (*c*) $\log_7 245$. (*d*) $\log_{13} 26$.

8. Prove that $\log_b a \cdot \log_a b = 1$.

9. Prove that

$$\log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2 \log_a [x + \sqrt{x^2 - 1}].$$

10. The velocity v in feet per second of a body that has fallen s feet is given by the formula $v = \sqrt{64.3\, s}$.

What is the velocity acquired by the body if it falls 45 ft. 7 in.?

11. Solve for x and y the equations: $2^x = 16^y$, $x + 4y = 4$.

CHAPTER VI

LOGARITHMIC COMPUTATION

45. Logarithmic Computation. In the last chapter a few examples of the use of logarithms in computation were given in connection with a four-place table. Such a table suffices for data and results accurate to four significant figures. When greater accuracy is desired we use a five-, six-, or seven-place table.

No subject is better adapted to illustrate the use of logarithmic computation than the solution of triangles, which we shall consider in some detail. Five-place tables and logarithmic solutions ordinarily are used at the same time, since both tend toward greater speed and accuracy.

46. Five-place Tables of Logarithms and Trigonometric Functions. The use of a five-place table of logarithms differs from that of a four-place table in the general use of so-called "interpolation tables" or "tables of proportional parts," to facilitate interpolation. Since the use of such tables of proportional parts is fully explained in every good set of tables, it is unnecessary to give such an explanation here. It will be assumed that the student has made himself familiar with their use.*

In the logarithmic solution of a triangle we nearly always need to find the logarithms of certain trigonometric functions. For example, if the angles A and B and the side a are given, we find the side b from the law of sines given in § 30,

$$b = \frac{a \sin B}{\sin A}.$$

* For this chapter, such a five-place table should be purchased. See, for example, THE MACMILLAN TABLES, which contain all the tables mentioned here with an explanation of their use.

To use logarithms we should then have to find $\log a$, $\log(\sin B)$ and $\log(\sin A)$. With only a table of natural functions and a table of logarithms at our disposal, we should have to find first $\sin A$, and then $\log \sin A$. For example, if $A = 36° 20'$, we would find $\sin 36° 20' = 0.59248$, and from this would find $\log \sin 36° 20' = \log 0.59248 = 9.77268 - 10$. This double use of tables has been made unnecessary by the direct tabulation of the logarithms of the trigonometric functions in terms of the angles. Such tables are called tables of logarithmic sines, logarithmic cosines, etc. Their use is explained in any good set of tables.

The following exercises are for the purpose of familiarizing the student with the use of such tables.

EXERCISES

1. Find the following logarithms:*

(*a*) $\log \cos 27° 40'.5$.
(*b*) $\log \tan 85° 20'.2$.
(*c*) $\log \sin 45° 40'.7$.
(*d*) $\log \text{ctn } 86° 53'.6$.
(*e*) $\log \cos 87° 6'.2$.
(*f*) $\log \cos 36° 53'.3$.

2. Find A, when

(*a*) $\log \sin A = 9.81632 - 10$.
(*b*) $\log \cos A = 9.97970 - 10$.
(*c*) $\log \tan A = 0.45704$.
(*d*) $\log \sin A = 9.78332 - 10$.
(*e*) $\log \text{ctn } \frac{1}{2} A = 0.70352$.
(*f*) $\log \tan \frac{1}{2} A = 9.94365 - 10$.

3. Find θ, if $\tan \theta = \dfrac{476.32 \times 89.710}{87325}$.

4. Given a triangle ABC, in which $\angle A = 32°$, $\angle B = 27°$, $a = 5.2$, find b by use of logarithms.

47. The Logarithmic Solution of Triangles. The effective use of logarithms in numerical computation depends largely on a proper arrangement of the work. In order to secure this, the arrangement should be carefully planned beforehand by constructing a *blank form*, which is afterwards filled in. Moreover, a practical computation is not complete until its accuracy has been checked. The blank form should provide also for a good check. Most computers find it advantageous to arrange

* Five-place logarithms are properly used when angles are measured to the nearest tenth of a minute. For accuracy to the nearest second, six places should be used.

the work in two columns, the one at the left containing the *given numbers* and the *computed results,* the one on the right containing the logarithms of the numbers each in the same horizontal line with its number. The work should be so arranged that every number or logarithm that appears is properly labeled; for it often happens that the same number or logarithm is used several times in the same computation and it should be possible to locate it at a glance when it is wanted.

The solution of triangles may be conveniently classified under four cases:

CASE I. *Given two angles and one side.*

CASE II. *Given two sides and the angle opposite one of the sides.*

CASE III. *Given two sides and the included angle.*

CASE IV. *Given the three sides.*

In each case it is desirable (1) to draw a figure representing the triangle to be solved with sufficient accuracy to serve as a rough check on the results; (2) to write out all the formulas needed for the solution and the check; (3) to prepare a blank form for the logarithmic solution on the basis of these formulas; (4) to fill in the blank form and thus to complete the solution.

We give a sample of a blank form under Case I; the student should prepare his own forms for the other cases.

48. Case I. Given Two Angles and One Side.

EXAMPLE. *Given:* $a=430.17$, $A=47°\ 13'.2$, $B=52°\ 29'.5$. (Fig. 37.)
To find: C, b, c.
Formulas:

$$C = 180° - (A + B),$$

$$b = \frac{a}{\sin A} \sin B,$$

$$c = \frac{a}{\sin A} \sin C.$$

CHECK (§ 36): $$\frac{c-b}{c+b} = \frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)}.$$

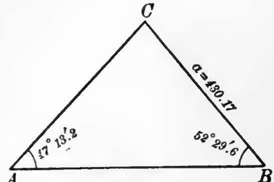


FIG. 37

The following is a convenient *blank form* for the logarithmic solution. The sign (+) indicates that the numbers should be added; the sign (−) indicates that the number should be subtracted from the one just above it.

Numbers				Logarithms	
$A =$	. . . . .				
$(+)B =$	. . . . .				
$A + B =$	. . . . .				
	179° 60′.0				
$C =$	. . . . .				
$a =$	. . . . .	(→)		. . . . .	
$\sin A = \sin$	. . .	(→)	(−)	. . . . .	
$a/\sin A$				. . . . .	
$\sin B = \sin$	. . .	(→)	(+)	. . . . .	
$b =$	. . . . .	(←)		. . . . .	
$a/\sin A$				. . . . .	
$\sin C = \sin$	. . .	(→)	(+)	. . . . .	
$c =$	. . . . .	(←)		. . . . .	

Check

$c - b =$	. . . . .	(→)		. . . . .	
$c + b =$	. . . . .	(→)	(−)	. . . . .	
					(1)
$C - B =$	. . . . .				
$C + B =$	. . . . .				(Logs (1) and (2)
$\tan \frac{1}{2}(C - B) = \tan$	. . .	(→)		. . . . .	should be equal
$\tan \frac{1}{2}(C + B) = \tan$	. . .	(→)	(−)	. . . . .	for check.)
					(2)

Filling in this blank form, we obtain the solution as follows.

Numbers				Logarithms
$A =$	47° 13′.2			
$B =$	52° 29′.6			
$A + B =$	99° 42′.8			
	179° 60′.0			
$C =$	80° 17′.2			
$a =$	430.17	(→)		2.63364
$\sin A =$	sin 47° 13′.2	(→)	(−)	9.86567 − 10
$a/\sin A$				2.76797
$\sin B =$	sin 52° 29′.6	(→)	(+)	9.89943 − 10
$b =$	464.94 *Ans.*	(←)		2.66740

$a/\sin A$			2.76797	
$\sin C = \sin 80° 17'.2$	(→)	(+)	9.99373 − 10	
$c = 577.70$ *Ans.*	(←)		2.76170	

CHECK

$c - b = 112.76$	(→)		2.05215	
$c + b = 1042.64$	(→)	(−)	3.01813	
			9.03402 − 10	CHECK *
$C - B = 27° 47'.6$				
$C + B = 132° 46'.8$				
$\tan \frac{1}{2}(C - B) = \tan 13° 53'.8$	(→)		9.39342 − 10	
$\tan \frac{1}{2}(C + B) = \tan 66° 23'.4$	(→)	(−)	0.35942	
			9.03400 − 10	

EXERCISES

Solve ~~and check~~ the following triangles ABC:

1. $a = 372.5,\ A = 25° 30',\ B = 47° 50'.$
2. $c = 327.85,\ A = 110° 52'.9,\ B = 40° 31'.7.$ *Ans.* $C = 28° 35'.4,$ $a = 640.11,\ b = 445.20.$
3. $a = 53.276,\ A = 108° 50'.0,\ C = 57° 13'.2.$
4. $b = 22.766,\ B = 141° 59'.1,\ C = 25° 12'.4.$
5. $b = 1000.0,\ B = 30° 30'.5,\ C = 50° 50'.8.$
6. $a = 257.7,\ A = 47° 25',\ B = 32° 26'.$

49. Case II. Given Two Sides and an Angle Opposite One of Them.

If A, a, b are given, B may be determined from the relation

$$\sin B = \frac{b \sin A}{a}. \tag{1}$$

If $\log \sin B = 0$, the triangle is a right triangle. Why?

If $\log \sin B > 0$, the triangle is impossible. Why?

If $\log \sin B < 0$, there are two possible values, B_1, B_2 of B, which are supplementary.

Hence there may be two solutions of the triangle. (See Example.)

No confusion need arise from the various possibilities if the corresponding figure is constructed and kept in mind.

It is desirable to go through the computation for log sin B

* A small discrepancy in the last figure need not cause concern. Why?

before making out the rest of the blank form, unless the data obviously show what the conditions of the problem actually are.

EXAMPLE 1. *Given:* $A = 46° 22'.2$, $a = 1.4063$, $b = 2.1048$. (Fig. 38.)
To find: B, C, c.
Formula: $\sin B = \frac{b \sin A}{a}$.

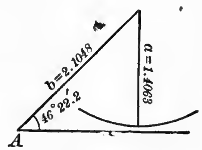


FIG. 38

Numbers			Logarithms
$b = 2.1048$	(→)		0.32321
$\sin A = \sin 46° 22'.2$	(→)	(+)	9.85962 − 10
$b \sin A$			0.18283
$a = 1.4063$	(→)	(−)	0.14808
$\sin B$	(←)		0.03475

Hence the triangle is impossible. Why?

EXAMPLE 2. *Given:* $a = 73.221$, $b = 101.53$, $A = 40° 22'.3$. (Fig. 39.)
To find: B, C, c.
Formula: $\sin B = \frac{b \sin A}{a}$.

Numbers			Logarithms
$b = 101.53$	(→)		2.00660
$\sin A = \sin 40° 22'.3$	(→)	(+)	9.81140 − 10
$b \sin A$			11.81800 − 10
$a = 73.221$	(→)	(−)	1.86464
$\sin B$			9.95336 − 10

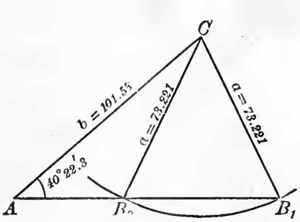


FIG. 39

The triangle is therefore possible and has two solutions (as the figure shows). We then proceed with the solution as follows:

We find one value B_1 of B from the value of log sin B. The other value B_2 of B is then given by $B_2 = 180° - B_1$.

Other formulas:

$$C = 180° - (A + B).$$

$$c = \frac{a \sin C}{\sin A}.$$

$$\text{CHECK: } \frac{c-b}{c+b} = \frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)}.$$

	Numbers			Logarithms
$\sin B$				$9.95336 - 10$
$B_1 =$	$63° 55'.2$			
	$179° 60'.0$			
$B_2 =$	$116° 4'.8$			
$A + B_1 =$	$104° 17'.5$			
	$179° 60'.0$			
$C_1 =$	$75° 42'.5$			
a		$(\rightarrow)$		1.86464
$\sin A$		$(\rightarrow)$	$(-)$	$9.81140 - 10$
$a/\sin A$				2.05324
$\sin C_1 =$	$\sin 75° 42'.5$	$(\rightarrow)$	$(+)$	$9.98634 - 10$
$c_1 =$	109.54	$(\leftarrow)$		2.03958
$c_1 - b =$	8.01	$(\rightarrow)$		0.90363
$c_1 + b =$	211.07	$(\rightarrow)$	$(-)$	2.32443
				$8.57920 - 10$ (CHECK)
$C_1 - B_1 =$	$11° 47'.3$			
$C_1 + B_1 =$	$139° 37'.7$			
$\tan \frac{1}{2}(C_1 - B_1) =$	$\tan 5° 53'.6$	$(\rightarrow)$		$9.01377 - 10$
$\tan \frac{1}{2}(C_1 + B_1) =$	$\tan 69° 48'.8$	$(\rightarrow)$		0.43455
				$8.57922 - 10$ (CHECK)

One solution of the triangle gives, therefore, $B = 63° 55'.2$, $C = 75° 42'.5$, $c = 109.54$.

To obtain the second solution, we begin with $B_2 = 116° 4'.8$. We find C_2 from $C_2 = 180° - (A + B_2)$; *i.e.* $C_2 = 23° 32'.9$. The rest of the computation is similar to that above and is left as an exercise.

EXERCISES

1. Show that, given A, a, b, if A is obtuse, or if A is acute and $a > b$, there cannot be more than one solution.

Solve the following triangles and check the solutions:

2. $a = 32.479$, $b = 40.176$, $A = 37° 25'.1$.

3. $b = 4168.2$, $c = 3179.8$, $B = 51° 21'.4$.

4. $a = 2.4621$, $b = 4.1347$, $B = 101° 37'.3$.

5. $a = 421.6$, $c = 532.7$, $A = 49° 21'.8$.

6. $a = 461.5$, $c = 121.2$, $C = 22° 31'.6$.

7. Find the areas of the triangles in Exs. 2–5.

50. Case III. Given Two Sides and the Included Angle.

Example. *Given:* $a = 214.17$, $b = 356.21$, $C = 62° 21'.4$. (Fig. 40.)

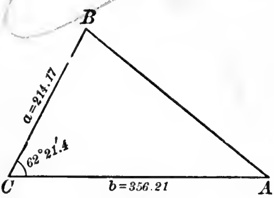


Fig. 40

To find: A, B, c.

Formulas:

$$\tan \tfrac{1}{2}(B - A) = \frac{b - a}{b + a} \tan \tfrac{1}{2}(B + A);$$

$$B + A = 180° - C = 117° 38'.6;$$

$$c = \frac{a \sin C}{\sin A}.$$

Numbers		Logarithms
$b - a = 142.04$	$(\rightarrow)$	2.15241
$b + a = 570.38$	$(\rightarrow)$	$(-)$ 2.75616
$(b - a)/(b + a)$		9.39625 − 10
$\tan \frac{1}{2}(B + A) = \tan 58° 49'.3$	$(\rightarrow)$	$(+)$ 0.21817
$\tan \frac{1}{2}(B - A) = \tan 22° 22'.2$	$(\leftarrow)$	9.61442 − 10
$\therefore A = 36° 27'.1$ *Ans.*		
$B = 81° 11'.5$ *Ans.*		
$a = 214.17$	$(\rightarrow)$	2.33076
$\sin A = \sin 36° 27'.1$	$(\rightarrow)$	$(-)$ 9.77389 − 10
$a/\sin A$		2.55687
$\sin C = \sin 62° 21'.4$	$(\rightarrow)$	$(+)$ 9.94736 − 10
$c = 319.32$ *Ans.*	$(\leftarrow)$	2.50423

Check by finding $\log (b/\sin B)$.

EXERCISES

Solve and check each of the following triangles:

1. $a = 74.801$, $b = 37.502$, $C = 63° 35'.5$.

2. $a = 423.84$, $b = 350.11$, $C = 43° 14'.7$.

3. $b = 275$, $c = 315$, $A = 30° 30'$.

4. $a = 150.17$, $c = 251.09$, $B = 40° 40'.2$.

5. $a = 0.25089$, $b = 0.30007$, $C = 42° 30' 20''$.

6. Find the areas of the triangles in Exs. 1–5.

51. Case IV. Given the Three Sides.

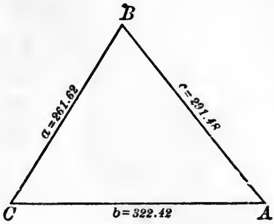


FIG. 41

EXAMPLE. *Given:* $a = 261.62$,
$b = 322.42$,
$c = 291.48$.
To find: A, B, C.

Formulas:

$$s = \tfrac{1}{2}(a + b + c).$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\tan \tfrac{1}{2} A = \frac{r}{s-a},\quad \tan \tfrac{1}{2} B = \frac{r}{s-b},\quad \tan \tfrac{1}{2} C = \frac{r}{s-c}. \qquad (\S\ 37)$$

CHECK: $A + B + C = 180°$.

Numbers			Logarithms
$a = 261.62$			
$b = 322.42$			
$c = 291.48$			
$2s = 875.52$			
$s = 437.76$			
$s - a = 176.14$	(→)		2.24586
$s - b = 115.34$	(→)		2.06198
$s - c = 146.28$	(→)	(+)	2.16518
			6.47302
$s = 437.76$ (CHECK).*	(→)	(−)	2.64124
r^2			3.83178
r			1.91589
$s - a$			2.24586
$\tan \frac{1}{2} A = \tan 25° 4'.1$	(←)		9.67003 − 10
r			1.91589
$s - b$			2.06198
$\tan \frac{1}{2} B = \tan 35° 32'.4$	(←)		9.85391 − 10
$r =$			1.91589
$s - c =$			2.16518
$\tan \frac{1}{2} C = \tan 29° 23'.4+$	(←)		9.75071 − 10

$A = 50° 8'.2$ *Ans.*
$B = 71° 4'.8$ *Ans.*
$C = 58° 46'.9$ *Ans.*
$179° 59'.9$ (CHECK.)

*By adding $s - a$, $s - b$, $s - c$.

$\operatorname{Tan} \frac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \frac{1}{2} C$

$\operatorname{Tan} \frac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \frac{1}{2} A$

EXERCISES

Solve and check each of the following triangles :

1. $a = 2.4169$, $b = 3.2417$, $c = 4.6293$.
2. $a = 21.637$, $b = 10.429$, $c = 14.221$.
3. $a = 528.62$, $b = 499.82$, $c = 321.77$.
4. $a = 2179.1$, $b = 3467.0$, $c = 5061.8$.
5. $a = 0.1214$, $b = 0.0961$, $c = 0.1573$.
6. Find the areas of the triangles in Exs. 1–5.
7. Find the areas of the inscribed circles of the triangles in Exs. 1–5.

OTHER LOGARITHMIC COMPUTATIONS

52. Interest and Annuities.

SIMPLE INTEREST.

Let the principal be represented by P
the interest on $1 for one year by r
the number of years by n
the amount of P for n years by A_n
Then the simple interest on P for a year is Pr
the amount of P for a year is $P + Pr = P(1+r)$,
the simple interest on P for n years is Pnr
the amount of P for n years is $A_n = P(1 + nr)$.

EXAMPLE. How long will it take $210, at 4 % simple interest, to amount to $298.20 ?

$$A_n = P(1 + nr) \qquad i.e. \qquad n = \frac{A_n - P}{Pr}.$$

Number		Logarithm	
$A_n - P = 88.20$	→	1.9455	
$Pr = 8.40$	→	0.9243	
$n = 10.5$	←	1.0212	10 yr. 6 mo. *Ans.*

COMPOUND INTEREST.

Let the original principal be P
and the rate of interest r
Then the amount A_1 at the end of the first year is

$$A_1 = P + Pr = P(1 + r),$$

the amount A_2 at the end of the second year is

$$A_2 = A_1(1+r) = P(1+r)^2,$$

the amount at the end of n years is

$$A_n = P(1+r)^n.$$

If the interest is compounded semiannually, $A_n = P\left(1+\frac{r}{2}\right)^{2n}$, if quarterly $A_n = P\left(1+\frac{r}{4}\right)^{4n}$, if q times a year $A_n = P\left(1+\frac{r}{q}\right)^{qn}$.

Since P in n years will amount to A_n, it is evident that P at the present time may be considered as equivalent in value to A due at the end of n years. Hence P is called the present worth of a given future sum A. Since

$$A_n = P(1+r)^n, \quad P = A_n(1+r)^{-n}.$$

EXAMPLE. In how many years will one dollar double itself at 4 % interest compounded annually?

$$A_n = P(1+r)^n \text{ or } \log\frac{A_n}{P} = n\log(1+r).$$

$$\therefore\ n = \frac{\log A_n - \log P}{\log(1+r)}.$$

Hence $$n = \frac{\log 2 - \log 1}{\log(1.04)} = \frac{0.3010}{0.0170} = 17.7.$$

17 yr. 9 mo. *Ans.*

ANNUITIES. An annuity is a fixed sum of money payable at equal intervals of time.

To find the present worth of an annuity of A dollars payable annually for n years, beginning one year hence, the rate of interest being r and the number of years n.

Since the present worth of the first payment is $A(1+r)^{-1}$, of the second $A(1+r)^{-2}$, etc., the present worth of the whole is

$$P = A[(1+r)^{-1} + (1+r)^{-2} + \cdots + (1+r)^{-n}].$$

The quantity in the brackets is a G. P. whose ratio is $(1+r)^{-1}$. Summing, we have

$$P = A\frac{(1+r)^{-1} - (1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right].$$

If the annuity is perpetual, *i.e.* n is infinite, the formula for present worth becomes $P = \frac{A}{r}$.

EXAMPLE. What should be paid for an annuity of $100 payable annually for 20 years, money being worth 4 % per annum?

$$P = \frac{100}{0.04}\left[1 - \frac{1}{(1.04)^{20}}\right].$$

By logarithms $(1.04)^{20} = 2.188.$

Therefore $P = \frac{100}{0.04}\left[1 - \frac{1}{2.188}\right] = 2500\left[\frac{1.188}{2.188}\right] = \$1358$, approximately.

53. Projectiles. Logarithms are used extensively in ballistic computations. [Ballistics is the science of the motion of a projectile.] The following is a very simple example of the type of problem considered.

The time of flight of a projectile (in vacuum) is given by the formula $T = \sqrt{\frac{2X\tan\phi}{g}}$ where X is the horizontal range in feet, ϕ is the angle of departure, and g is the acceleration due to gravity in feet per second per second [$g = 32.2$]. If it is known that the range is 3000 yd. and that the angle of departure is 30° 20′, find the time of flight.

$$T = \sqrt{\frac{2X\tan\phi}{g}}.$$

Numbers		Logarithms
$2X = 18000$	→	4.2553
tan 30° 20′	→	9.7673 − 10
		4.0226
32.2	→	1.5079.
		2)2.5147
18.09	←	1.2574 $T = 18.09$ seconds. *Ans.*

EXERCISES

1. Find the amount of $500 in 10 years at 4 per cent compound interest, compounded semiannually.

2. In how many years will a sum of money double itself at 5 per cent interest compounded annually? semiannually?

3. A thermometer bulb at a temperature of 20° C. is exposed to the air for 15 seconds, in which time the temperature drops 4 degrees. If the law of cooling is given by the formula $\theta = \theta_0 e^{-bt}$, where θ is the final temperature, θ_0 the initial temperature, e the natural base of logarithms, and t the time in seconds, find the value of b.

4. The stretch s of a brass wire when a weight m is hung at its free end is given by the formula

$$s = \frac{mgl}{\pi r^2 k},$$

where m is the weight applied in grams, $g = 980$, l is the length of the wire in centimeters, r is the radius of the wire in centimeters, and k is a constant. If $m = 844.9$ grams, $l = 200.9$ centimeters, $r = 0.30$ centimeter when $s = 0.056$, find k.

5. The crushing weight P in pounds of a wrought-iron column is given by the formula

$$P = 299{,}600 \frac{d^{3.55}}{l^2},$$

where d is the diameter in inches and l is the length in feet. What weight will crush a wrought-iron column 10 feet long and 2.7 inches in diameter?

6. The number n of vibrations per second made by a stretched string is given by the relation

$$n = \frac{1}{2\,l}\sqrt{\frac{Mg}{m}},$$

where l is the length of the string in centimeters, M is the weight in grams that stretches the string, m the weight in grams of one centimeter of the string, and $g = 980$. Find n when $M = 5467.9$ grams, $l = 78.5$ centimeters, $m = 0.0065$ gram.

7. The time t of oscillation of a pendulum of length l centimeters is given by the formula

$$t = \pi\sqrt{\frac{l}{980}}.$$

Find the time of oscillation of a pendulum 73.27 centimeters in length.

8. The weight w in grams of a cubic meter of aqueous vapor saturated at 17° C. is given by the formula

$$w = \frac{1293 \times 12.7 \times 5}{(1 + \frac{17}{273})(760 \times 8)}.$$

Compute w.

54. The Logarithmic Scale. An arithmetic scale in which the segments from the origin are proportional to the logarithms of 1, 2, 3, etc., is called a logarithmic scale. Such a scale is given in Fig. 42.

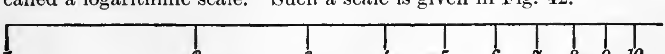


FIG. 42

55. The Slide Rule. The slide rule consists of a rule along the center of which a slip of the same material slides in a groove. Along the

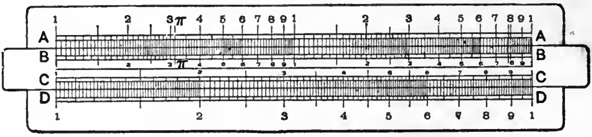


FIG. 43

upper edge of the groove are engraved two logarithmic scales, *A* and *B*, that are identical. Along the lower edge are also two identical logarithmic scales, *C* and *D*, in which the unit is twice that in scales *A* and *B*. Since the segments represent the logarithms of the numbers found in the scale, the operation of adding the segments is equivalent to multiplying the

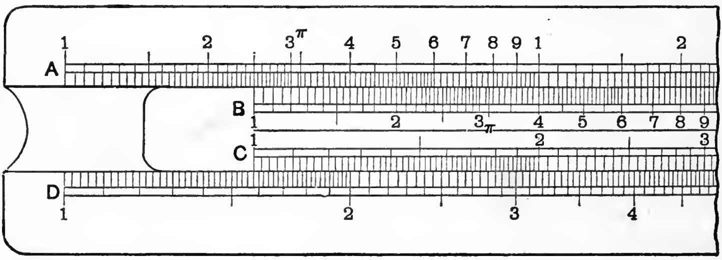


FIG. 44

corresponding numbers. Thus in Fig. 44 the point marked 1 on scale *B* is set opposite the point marked 2.5 on scale *A*. The point marked 4 on scale *B* will be opposite the point marked 10 on scale *A*, *i.e.* $2.5 \times 4 = 10$. Similarly we read $2.5 \times 3.2 = 8$, $2.5 \times 2.5 = 6.25$. Other multiplications can be performed in an analogous manner.

Division can be performed by reversing the operation. Thus in Fig. 44 every number of scale *B* is the result of dividing the number above it by 2.5. Thus we read $7.2 \div 2.5 = 2.9$ approximately.

Since scales *C* and *D* are twice as large as scales *A* and *B*, it follows that the numbers in these scales are the square roots of the numbers opposite to them in scales *A* and *B*. Conversely the numbers on scales *A* and *B* are the squares of the numbers opposite them on scales *C* and *D*. Moreover the scales *C* and *D* can be used for multiplying and dividing, but the range of numbers is not so large.

For a more complete discussion of the use of a slide rule consult the book of instructions published by any of the manufacturers of slide rules, where also exercises will be found for practice.

CHAPTER VII

TRIGONOMETRIC RELATIONS

56. Radian Measure. In certain kinds of work it is more convenient in measuring angles to use, instead of the degree, a unit called the radian. A ***radian*** is defined as the angle at the center of a circle whose subtended arc is equal in length to the radius of the circle (Fig. 45). Therefore, if an angle θ at the center of a circle of radius r units subtends an arc of s units, the measure of θ in radians is

(1) $$\theta = \frac{s}{r}.$$

Since the length of the whole circle is $2\pi r$, it follows that

$$\frac{2\pi r}{r} = 2\pi \text{ radians} = 360°,$$

or

(2) π ***radians*** **= 180°.**

Therefore,

$$1 \text{ radian} = \frac{180°}{\pi} = 57° \ 17' \ 45'' \text{ (approximately).}$$

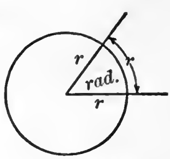


FIG. 45

It is important to note that the radian * as defined is a constant angle, *i.e.* it is the same for all circles, and can therefore be used as a unit of measure.

From relation (2) it follows that to convert radians into degrees it is only necessary to multiply the number of radians by $180/\pi$, while to convert degrees into radians we multiply the number of degrees by $\pi/180$. Thus 45° is $\pi/4$ radians; $\pi/2$ radians is 90°.

* The symbol ^{r} is often used to denote radians. Thus 2^r stands for 2 radians, π^r for π radians, etc. When the angle is expressed in terms of π (the radian being the unit), it is customary to omit ^{r}. Thus, when we refer to an angle π, we mean an angle of π radians. When the word radian is omitted, it should be mentally supplied in order to avoid the error of supposing π means 180. Here, as in geometry, $\pi = 3.14159. . . .$

57. The Length of Arc of a Circle. From relation (1), § 56, it follows that

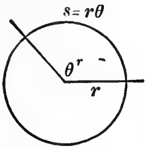


FIG. 46

$$s = r\theta.$$

That is (Fig. 46), if a central angle is measured in radians, and if its intercepted arc and the radius of the circle are measured in terms of the *same unit*, then

length of arc = radius × central angle in radians.

EXERCISES

1. Express the following angles in radians:

$$25°,\ 145°,\ 225°,\ 300°,\ 270°,\ 450°,\ 1150°.$$

2. Express in degrees the following angles:

$$\frac{\pi}{4},\ -\frac{7\pi}{6},\ \frac{5\pi}{6},\ 3\pi,\ \frac{5\pi}{4}.$$

3. A circle has a radius of 20 inches. How many radians are there in an angle at the center subtended by an arc of 25 inches? How many degrees are there in this same angle? *Ans.* $\frac{5}{4}^r$; 71° 37′ approx.

4. Find the radius of a circle in which an arc 12 inches long subtends an angle of 35°.

5. The minute hand of a clock is 4 feet long. How far does its extremity move in 22 minutes?

6. In how many hours is a point on the equator carried by the rotation of the earth on its axis through a distance equal to the diameter of the earth?

7. A train is traveling at the rate of 10 miles per hour on a curve of half a mile radius. Through what angle has it turned in one minute?

8. A wheel 10 inches in diameter is belted to a wheel 3 inches in diameter. If the first wheel rotates at the rate of 5 revolutions per minute, at what rate is the second rotating? How fast must the former rotate in order to produce 6000 revolutions per minute in the latter?

58. Angular Measurement in Artillery Service. The divided circles by means of which the guns of the United States Field Artillery are aimed are graduated neither in degrees nor in radians, but in units called *mils*. The mil is defined as an angle subtended by an arc of $\frac{1}{6400}$ of the circumference, and is therefore equal to

$$\frac{2\pi}{6400} = \frac{3.1416}{3200} = 0.00098175 = (0.001 - 0.00001825) \text{ radian.}$$

The mil is therefore approximately one thousandth of a radian. (Hence its name.)*

Since (§ 57)

length of arc = radius × central angle in radians,

it follows that we have *approximately*

$$\text{length of arc} = \frac{\text{radius}}{1000} \times \text{central angle in mils};$$

i.e. length of arc in yards = (radius in thousands of yards) · (angle in mils). The error here is about 2 %.

FIG. 47

EXAMPLE 1. A battery occupies a front of 60 yd. If it is at 5500 yd. range, what angle does it subtend (Fig. 47)? We have, evidently,

$$\text{angle} = \frac{60}{5.5} = 11 \text{ mils.}$$

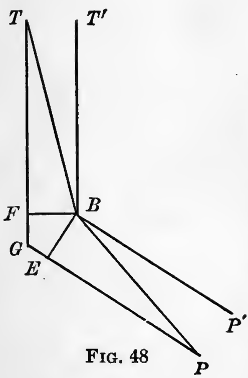


FIG. 48

EXAMPLE 2. *Indirect Fire.* † A battery posted with its right gun at G is to open fire on a battery at a point T, distant 2000 yd. and invisible from G (Fig. 48). The officer directing the fire takes post at a point B from which both the target T and a church spire P, distant 3000 yd. from G, are visible. B is 100 yd. at the right of the line GT and 120 yd. at the right of the line GP and the officer finds by measurement that the angle PBT contains 3145 mils. In order to train the gun on the target the gunner must set off the angle PGT on the sight of the piece and then move the gun

* To give an idea of the value in mils of certain angles the following has been taken from the *Drill Regulations for Field Artillery* (1911), p. 164:

"Hold the hand vertically, palm outward, arm fully extended to the front. Then the angle subtended by the

width of thumb is	40 mils
width of first finger at second joint is	40 mils
width of second finger at second joint is	40 mils
width of third finger at second joint is	35 mils
width of little finger at second joint is	30 mils
width of first, second, and third fingers at second joint is .	115 mils

These are average values."

† The limits of the text preclude giving more than a single illustration of the problems arising in artillery practice. For other problems the student is referred to the *Drill Regulations for Field Artillery* (1911), pp. 57, 61, 150–164; and to ANDREWS, *Fundamentals of Military Service*, pp. 153–159, from which latter text the above example is taken.

until the spire P is visible through the sight. When this is effected, the gun is aimed at T.

Let F and E be the feet of the perpendiculars from B to GT and GP respectively, and let BT' and BP' be the parallels to GT and GP that pass through B. Then, evidently, if the officer at B measures the angle PBT, which would be used instead of angle PGT were the gun at B instead of at G, and determines the angles $TBT' = FTB$ and $PBP' = EPB$, he can find the angle PGT from the relation

$$PGT = P'BT' = PBT - TBT' - PBP'.$$

Now $$\tan FTB = \frac{FB}{TF}, \quad \tan EPB = \frac{EB}{PE}.$$

Furthermore if FTB and EPB are small angles, *i.e.* if FB and EB are small compared with GT and GP respectively, the radian measure of the angle is approximately equal to the tangent of the angle. Why? Hence we have

$$\left.\begin{aligned} FTB &= \tan FTB = \frac{FB}{GT} \\ EPB &= \tan EPB = \frac{EB}{GP} \end{aligned}\right\} \text{approximately.}$$

Therefore $$TBT' = FTB = \frac{100}{2000} \text{ radians} = 50 \text{ mils},$$

$$PBP' = EPB = \frac{120}{3000} \text{ radians} = 40 \text{ mils}.$$

Hence $$\begin{aligned} PGT &= PBT - TBT' - PBP' \\ &= 3145 - 50 - 40 \\ &= 3055 \text{ mils}, \end{aligned}$$

which is the angle to be set off on the sight of the gun.

Hence from the situation indicated in Fig. 48 we have the following rule:

(1) Measure in mils the angle PBT from the aiming point P to the target T as seen at B.

(2) Measure or estimate the offsets FB and EB in yards, the range GT and the distance GP of the aiming point P in *thousands* of yards.

(3) Compute in mils the offset angles by means of the relations

$$\begin{aligned} TBT' &= FTB, \\ PBP' &= EPB, \\ TBT' &= \frac{FB}{GT}. \\ PBP' &= \frac{EB}{GP}. \end{aligned}$$

(4) Then the angle of deflection PGT is equal to the angle PBT diminished by the sum of the offset angles.

EXERCISES

1. A battery occupies a front of 80 yd. It is at 5000 yd. range. What angle does it subtend?

2. In Fig. 48 suppose $PBT = 3000$ mils, $FB = 200$ yd., $GT = 3000$ yd., $EB = 150$ yd., $GP = 4000$ yd. Find the number of mils in PGT.

3. A battery at a point G is ordered to take a masked position and be ready to fire on an indicated hostile battery at a point T whose range is known to be 2100 yd. The battery commander finds an observing station B, 200 yd. at the right and on the prolongation of the battery front, and 175 yd. at the right of PG. An aiming point P, 5900 yd. in the rear, is found, and PBT is found to be 2600 mils. Find PGT.

4. A battery at a point G is to fire on an invisible object at a point T whose range is known to be 2000 yd. A battery commander finds an observing station B, 100 yd. at the right of GT and 150 yd. at the right of GP. The aiming point P is 1500 yd. in front and to the left of GT. The angle TBP contains 1200 mils. Find PGT.

59. The Sine Function. Let us trace in a general way the variation of the function $\sin\theta$ as θ increases from 0° to 360°. For this purpose it will be convenient to think of the distance r as constant, from which it follows that the locus of P is a circle. When $\theta = 0°$, the point P lies on the x-axis and hence the ordinate is 0, *i.e.* $\sin 0° = 0/r = 0$. As θ increases to 90°, the ordinate increases until 90° is reached, when it becomes equal to r. Therefore, $\sin 90° = r/r = 1$. As θ increases from 90° to 180°, the ordinate decreases until 180° is reached, when it becomes 0. Therefore $\sin 180° = 0/r =$ As θ increases from 180° to 270°, the ordinate of P continually decreases algebraically and reaches its smallest algebraic value when $\theta = 270°$. In this position the ordinate is $-r$ and $\sin 270° = -r/r = -1$. When θ enters the fourth quadrant, the ordinate of P increases (algebraically) until the angle reaches 360°, when the ordinate becomes 0.

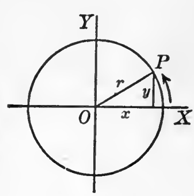


FIG. 49

Hence, $\sin 360° = 0$. It then appears that:

as θ increases from 0° to 90°, $\sin\theta$ increases from 0 to 1;
as θ increases from 90° to 180°, $\sin\theta$ decreases from 1 to 0;
as θ increases from 180° to 270°, $\sin\theta$ decreases from 0 to -1;
as θ increases from 270° to 360°, $\sin\theta$ increases from -1 to 0.

It is evident that the function $\sin\theta$ repeats its values in the same order no matter how many times the point P moves around the circle. We express this fact by saying that the function $\sin\theta$ is **periodic** and has a *period* of 360°. In symbols this is expressed by the equation

$$\sin[\theta + n \cdot 360°] = \sin\theta,$$

where n is any positive or negative integer.

The variation of the function $\sin\theta$ is well shown by its graph. To construct this graph proceed as follows: Take a system of rectangular axes and construct a circle of unit radius

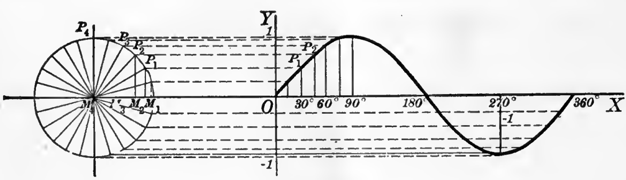


FIG. 50

with its center on the x-axis (Fig. 50). Let angle $XM_4P = \theta$. Then the values of $\sin\theta$ for certain values of θ are shown in the unit circle as the ordinates of the end of the radius drawn at an angle θ.

θ	0	30°	45°	60°	90°	...
$\sin\theta$	0	M_1P_1	M_2P_2	M_3P_3	M_4P_4	...

Now let the number of degrees in θ be represented by distances measured along OX. At a distance that represents 30° erect a perpendicular equal in length to $\sin 30°$; at a distance that represents 60° erect one equal in length to sin 60°, etc. Through the points O, P_1, P_2, ⋯ draw a smooth curve; this curve is the graph of the function sin θ.

If from any point P on this graph a perpendicular PQ is drawn to the x-axis, then QP represents the sine of the angle represented by the segment OQ.

Since the function is periodic, the complete graph extends indefinitely in both directions from the origin (Fig. 51).

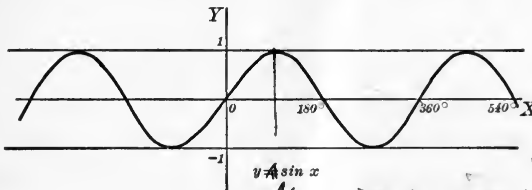


FIG. 51

60. The Cosine Function. By arguments similar to those used in the case of the sine function we may show that:
as θ increases from 0° to 90°, the cos θ decreases from 1 to 0;
as θ increases from 90° to 180°, the cos θ decreases from 0 to -1;
as θ increases from 180° to 270°, the cos θ increases from -1 to 0;
as θ increases from 270° to 360°, the cos θ increases from 0 to 1.

The graph of the function is readily constructed by a method

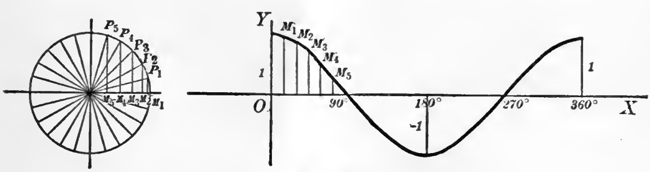


FIG. 52

similar to that used in the case of the sine function. This is illustrated in Fig. 52.

The complete graph of the cosine function, like that of the sine function, will extend indefinitely from the origin in both directions (Fig. 53). Moreover $\cos\theta$, like $\sin\theta$, is *periodic* and has a *period* of 360°, *i.e.*

$$\cos[\theta + n \cdot 360°] = \cos\theta,$$

where n is any positive or negative integer.

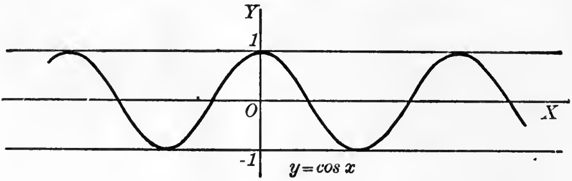


FIG. 53

61. The Tangent Function. In order to trace the variation of the tangent function, consider a circle of unit radius with its center at the origin of a system of rectangular axes (Fig. 54). Then construct the tangent to this circle at the point $M(1, 0)$ and let P denote any point on this tangent line. If angle $MOP = \theta$, we have $\tan\theta = MP/OM = MP/1 = MP$, *i.e.* the line MP represents $\tan\theta$.

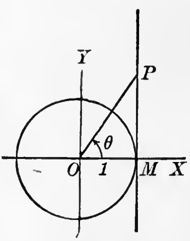


FIG. 54

Now when $\theta = 0°$, MP is 0, *i.e.* $\tan 0°$ is 0. As the angle θ increases, $\tan\theta$ increases. As θ approaches 90° as a limit, MP becomes infinite, *i.e.* $\tan\theta$ becomes larger than any number whatever.

At 90° *the tangent is undefined.* It is sometimes convenient to express this fact by writing

$$\tan 90° = \infty.$$

However we must remember that this is *not a definition* for tan 90°, for ∞ is not a number. This is merely a short way of saying that *as* θ *approaches* 90° *tan* θ *becomes infinite* and that at 90° $\tan\theta$ is undefined.

Thus far we have assumed θ to be an acute angle approaching 90° as a limit. Now let us start with θ as an obtuse angle and let it decrease towards 90° as a limit. In Fig. 55 the line MP' (which is here negative in direction) represents $\tan\theta$. Arguing precisely as we did before, it is seen that as the angle θ approaches 90° as a limit, $\tan\theta$ again increases in magnitude beyond all bounds, *i.e.* becomes infinite, remaining, however, always negative.

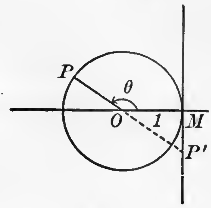


Fig. 55

We then have the following results.

(1) When θ is acute and increases towards 90° as a limit, $\tan\theta$ always remains positive but becomes infinite. At 90° $\tan\theta$ is undefined.

(2) When θ is obtuse and decreases towards 90° as a limit, $\tan\theta$ always remains negative but becomes infinite. At 90° $\tan\theta$ is undefined.

It is left as an exercise to finish tracing the variation of the tangent function as θ varies from 90° to 360°. Note that tan 270°, like tan 90°, is undefined. In fact $\tan n \cdot 90°$ is undefined, if n is any odd integer.

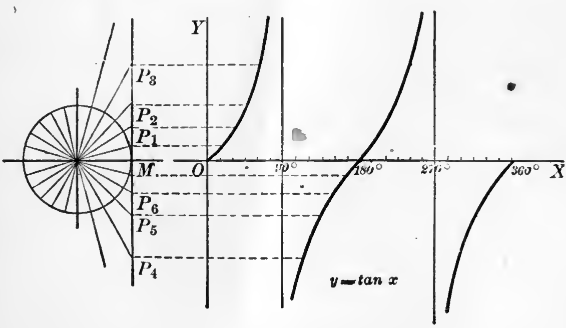


Fig. 56

To construct the graph of the function $\tan\theta$ we proceed along lines similar to those used in constructing the graph of $\sin\theta$ and $\cos\theta$. The following table together with Fig. 56 illustrates the method.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°
$\tan\theta$	0	MP_1	MP_2	MP_3	undefined	MP_4	MP_5	MP_6	$MP_7=0$	MP_1

It is important to notice that $\tan\theta$, like $\sin\theta$ and $\cos\theta$, is *periodic*, but its *period* is 180°. That is

$$\tan(\theta + n \cdot 180°) = \tan\theta,$$

where n is any positive or negative integer.

EXERCISES

1. What is meant by the period of a trigonometric function?

2. What is the period of $\sin\theta$? $\cos\theta$? $\tan\theta$?

3. Is $\sin\theta$ defined for all angles? $\cos\theta$?

4. Explain why $\tan\theta$ is undefined for certain angles. Name four angles for which it is undefined. Are there any others?

5. Is $\sin(\theta + 360°) = \sin\theta$?

6. Is $\sin(\theta + 180°) = \sin\theta$?

7. Is $\tan(\theta + 180°) = \tan\theta$?

8. Is $\tan(\theta + 360°) = \tan\theta$?

Draw the graphs of the following functions and explain how from the graph you can tell the period of the function:

9. $\sin\theta$.

10. $\cos\theta$.

11. $\tan\theta$.

12. $\csc\theta$.

13. $\sec\theta$.

14. $\text{ctn}\,\theta$.

Verify the following statements:

15. $\sin 90° + \sin 270° = 0$.

16. $\cos 90° + \sin 0° = 0$.

17. $\tan 180° + \cos 180° = -1$.

18. $\cos 180° + \sin 180° = -1$.

19. $\tan 360° + \cos 360° = 1$.

20. $\cos 90° + \tan 180° - \sin 270° = 1$.

21. Draw the graphs of the functions $\sin\theta$, $\cos\theta$, $\tan\theta$, making use of a table of natural functions. See p. 112.

22. Draw the curves $y = 2\sin\theta$; $y = 2\cos\theta$; $y = 2\tan\theta$.

23. Draw the curve $y = \sin\theta + \cos\theta$.

24. From the graphs determine values of θ for which $\sin\theta = \frac{1}{2}$; $\sin\theta = 1$; $\tan\theta = 1$; $\cos\theta = \frac{1}{2}$; $\cos\theta = 1$.

Here for test

62. The Trigonometric Functions of $-\theta$. Draw the angles θ and $-\theta$, where OP is the terminal line of θ and OP' is the terminal line of $-\theta$. Figure 57 shows an angle θ in each of

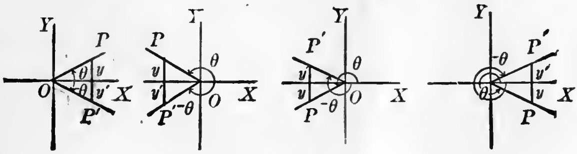


FIG 57

the four quadrants. We shall choose $OP = OP'$ and (x, y) as the coördinates of P and (x', y') as the coördinates of P'. In all four figures

$$x' = x,\ y' = -y,\ r' = r.$$

Hence

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta.$$

Also,

$$\csc(-\theta) = -\csc\theta;\ \sec(-\theta) = \sec\theta;\ \text{ctn}(-\theta) = -\text{ctn}\,\theta.$$

The above results can be stated as follows: The functions of $-\theta$ equal numerically the like named functions of θ. The algebraic sign, however, will be opposite except for the cosine and secant.

EXAMPLE. $\sin -10° = -\sin 10°$, $\cos -10° = \cos 10°$, $\tan -10° = -\tan 10°$.

63. The Trigonometric Functions of $180° + \theta$. Similarly, the following relations hold:

$\sin(180° + \theta) = -\sin\theta$, $\csc(180° + \theta) = -\csc\theta$,
$\cos(180° + \theta) = -\cos\theta$, $\sec(180° + \theta) = -\sec\theta$,
$\tan(180° + \theta) = \tan\theta$, $\text{ctn}(180° + \theta) = \text{ctn}\,\theta$.

The proof is left as an exercise.

64. Summary. An inspection of the results of §§ 27–28, 62–63 shows:

1. *Each function of* $-\theta$ *or* $180° \pm \theta$ *is equal in absolute value* (but not always in sign) *to the same function of* θ.

2. *Each function of* $90° - \theta$ *is equal in magnitude and in sign to the corresponding co-function of* θ.

These principles enable us to find the value of any function of any angle in terms of a function of a positive acute angle (not greater than 45° if desired) as the following examples show.

EXAMPLE 1. Reduce $\cos 200°$ to a function of an angle less than 45°.

Since 200° is in the third quadrant, $\cos 200°$ is negative. Hence $\cos 200° = -\cos 20°$. Why?

EXAMPLE 2. Reduce $\tan 260°$ to a function of an angle less than 45°.

Since 260° is in the third quadrant, $\tan 260°$ is positive. Hence $\tan 260° = \tan 80° = \text{ctn}\, 10°$ (§ 27).

EXAMPLE 3. Reduce $\sin(-210°)$ to a function of a positive angle less than 45°.

From § 62 we know $\sin -210° = -\sin 210°$.

Considering the positive angle 210°, we have

$$\sin -210° = -\sin 210° = -[-\sin 30°] = \sin 30°.$$

EXERCISES

Reduce to a function of an angle not greater than 45°:

1. $\sin 163°$.
2. $\cos(-110°)$.
   Ans. $-\sin 20°$.
3. $\sec(-265°)$.
4. $\tan 428°$.
5. $\csc 901°$.
6. $\text{ctn}(-1215°)$.
7. $\tan 840°$.
8. $\sin 510°$.

Find without the use of tables the values of the following functions:

9. $\cos 570°$.
10. $\sin 330°$.
11. $\tan 390°$.
12. $\sin 420°$.
13. $\cos 150°$.
14. $\tan 300°$.

Reduce the following to functions of positive acute angles:

15. $\sin 250°$.
   Ans. $-\sin 70°$ or $-\cos 20°$.
16. $\cos 158°$.
17. $\tan(-389°)$.
18. $\sec(-245°)$.
19. $\csc(-321°)$.
20. $\sin 269°$.

Prove the following relations from a figure:

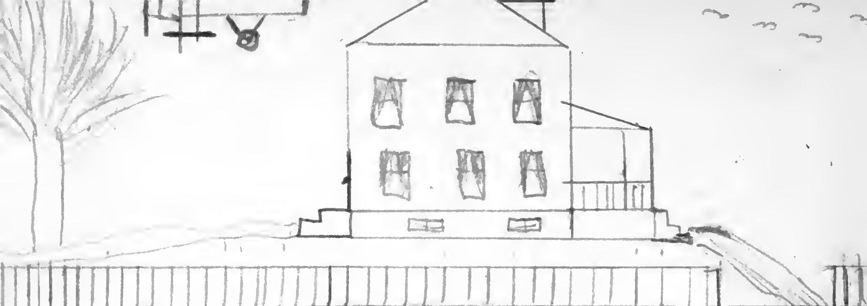
(*a*)
$\sin(90^\circ + \theta) = \cos\theta.$
$\cos(90^\circ + \theta) = -\sin\theta.$
$\tan(90^\circ + \theta) = -\operatorname{ctn}\theta.$
$\csc(90^\circ + \theta) = \sec\theta.$
$\sec(90^\circ + \theta) = -\csc\theta.$
$\operatorname{ctn}(90^\circ + \theta) = -\tan\theta.$

(*b*)
$\sin(180^\circ - \theta) = \sin\theta.$
$\cos(180^\circ - \theta) = -\cos\theta.$
$\tan(180^\circ - \theta) = -\tan\theta.$
$\csc(180^\circ - \theta) = \csc\theta.$
$\sec(180^\circ - \theta) = -\sec\theta.$
$\operatorname{ctn}(180^\circ - \theta) = -\operatorname{ctn}\theta.$

(*c*)
$\sin(180^\circ + \theta) = -\sin\theta.$
$\cos(180^\circ + \theta) = -\cos\theta.$
$\tan(180^\circ + \theta) = \tan\theta.$
$\csc(180^\circ + \theta) = -\csc\theta.$
$\sec(180^\circ + \theta) = -\sec\theta.$
$\operatorname{ctn}(180^\circ + \theta) = \operatorname{ctn}\theta.$

(*d*)
$\sin(270^\circ - \theta) = -\cos\theta.$
$\cos(270^\circ - \theta) = -\sin\theta.$
$\tan(270^\circ - \theta) = \operatorname{ctn}\theta.$
$\csc(270^\circ - \theta) = -\sec\theta.$
$\sec(270^\circ - \theta) = -\csc\theta.$
$\operatorname{ctn}(270^\circ - \theta) = \tan\theta.$

(*e*)
$\sin(270^\circ + \theta) = -\cos\theta.$
$\cos(270^\circ + \theta) = \sin\theta.$
$\tan(270^\circ + \theta) = -\operatorname{ctn}\theta.$
$\csc(270^\circ + \theta) = -\sec\theta.$
$\sec(270^\circ + \theta) = \csc\theta.$
$\operatorname{ctn}(270^\circ + \theta) = -\tan\theta.$



CHAPTER VIII

TRIGONOMETRIC RELATIONS (*Continued*)

65. Trigonometric Equations. An identity, as we have seen (§ 26), is an equality between two expressions which is satisfied for all values of the variables for which both expressions are defined. If the equality is not satisfied for all values of the variables for which each side is defined, it is called a conditional equality, or simply an equation. Thus $1 - \cos\theta = 0$ is true only if $\theta = n \cdot 360°$, where n is an integer. To solve a trigonometric equation, *i.e.* to find the values of θ for which the equality is true, we usually proceed as follows.

1. Express all the trigonometric functions involved in terms of one trigonometric function of the *same* angle.

2. Find the value (or values) of this function by ordinary algebraic methods.

3. Find the angles between 0° and 360° which correspond to the values found. These angles are called *particular solutions.*

4. Give the general solution by adding $n \cdot 360°$, where n is any integer, to the particular solutions.

Example 1. Find θ when $\sin\theta = \frac{1}{2}$.

The particular solutions are 30° and 150°. The general solutions are $30° + n \cdot 360°$, $150° + n \cdot 360°$.

Example 2. Solve the equation $\tan\theta \sin\theta - \sin\theta = 0$.

Factoring the expression, we have $\sin\theta(\tan\theta - 1) = 0$. Hence we have $\sin\theta = 0$, or $\tan\theta - 1 = 0$. Why?

The particular solutions are therefore 0°, 180°, 45°, 225°. The general solutions are $n \cdot 360°$, $180° + n \cdot 360°$, $45° + n \cdot 360°$, $225° + n \cdot 360°$.

EXERCISES

Give the particular and the general solutions of the following equations:

1. $\sin\theta = \frac{\sqrt{3}}{2}$.
2. $\sin\theta = -\frac{\sqrt{3}}{2}$.
3. $\cos\theta = \frac{\sqrt{3}}{2}$.
4. $\cos\theta = -\frac{\sqrt{3}}{2}$.
5. $\tan\theta = -1$.
6. $\operatorname{ctn}\theta = 1$.
7. $\sec\theta = 2$.
8. $\tan\theta = 0$.
9. $\sec^2\theta = 2$.
10. $\sin^2\theta = \frac{1}{2}$.
11. $\cos\theta = -\frac{1}{2}$.
12. $\csc^2\theta = \frac{4}{3}$.
13. $4\sin\theta - 3\csc\theta = 0$.
14. $2\sin\theta\cos^2\theta = \sin\theta$.
15. $\cos\theta + \sec\theta = \frac{5}{2}$.
16. $2\sin\theta = \tan\theta$. *Ans.* Particular solutions: 0°, 180°, 60°, 300°.
17. $3\sin\theta + 2\cos\theta = 2$.
18. $2\cos^2\theta - 1 = 1 - \sin^2\theta$.

66. Inverse Trigonometric Functions. The equation

$$x = \sin y \tag{1}$$

may be read:

y is an angle whose sine is equal to x,

a statement which is usually written in the contracted form

$$y = \text{arc}\sin x.\text{*} \tag{2}$$

For example, $x = \sin 30°$ means that $x = \frac{1}{2}$, while $y = \text{arc}\sin\frac{1}{2}$ means that $y = 30°, 150°$, or in general (n being an integer),

$$30° + n\cdot 360°;\ 150° + n\cdot 360°.$$

Since the sine is never greater than 1 and never less than -1, it follows that $-1 \leqq x \leqq 1$. It is evident that there is an unlimited number of values of $y = \text{arc}\sin x$ for a given value of x in this interval.

We shall now define the *principal value* $\text{Arc}\sin x$ † of $\text{arc}\sin x$, distinguished from $\text{arc}\sin x$ by the use of the capital A, to be

* Sometimes written $y = \sin^{-1} x$. Here -1 is not an algebraic exponent, but merely a part of a functional symbol. When we wish to raise $\sin x$ to the power -1, we write $(\sin x)^{-1}$.

† Sometimes written $\text{Sin}^{-1} x$, distinguished from $\sin^{-1} x$ by the use of the capital S.

the numerically smallest angle whose sine is equal to x. This function like arc sin x is defined only for those values of x for which

$$-1 \leqq x \leqq 1.$$

The difference between arc sin x and Arc sin x is well illustrated by means of their graph. It is evident that the graph of $y = \text{arc}\sin x$, *i.e.* $x = \sin y$ is simply the sine curve with the rôle of the x and y axes interchanged. (See Fig. 58.) Then for every admissible value of x, there is an unlimited number of values of y; namely, the ordinates of all the points $P_1, P_2, \cdots$, in which a line at a distance x and parallel to the y-axis intersects the curve. The single-valued function Arc sin x is represented by the part of the graph between M and N.

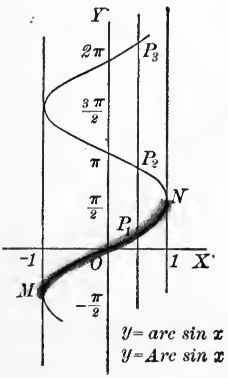


FIG. 58

Similarly arc cos x, defined as "an angle whose cosine is x," has an unlimited number of values for every admissible value of $x\,(-1 \leqq x \leqq 1)$ We shall define the principal value Arc cos x as the smallest positive angle whose cosine is x. That is,

$$0 \leqq \text{Arc}\cos x \leqq \pi.$$

Figure 59 represents the graph of $y = \text{arc}\cos x$, and the portion of this graph between M and N represents Arc cos x.

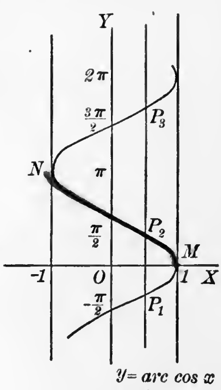


FIG. 59

Similarly we write $x = \tan y$ as $y = \text{arc}\tan x$, and in the same way we define the symbols arc ctn x; arc sec x; arc csc x. The principal values of all the inverse trigonometric functions are given in the following table.

$y =$	Arc sin x	Arc cos x	Arc tan x
Range of x	$-1 \leqq x \leqq 1$	$-1 \leqq x \leqq 1$	all real values
Range of y	$-\frac{\pi}{2}$ to $\frac{\pi}{2}$	0 to π	$-\frac{\pi}{2}$ to $\frac{\pi}{2}$
x positive	1st Quad.	1st Quad.	1st Quad.
x negative	4th Quad.	2d Quad.	4th Quad.
	Arc ctn x	Arc sec x	Arc csc x
Range of x	all values	$x \geqq 1$ or $x \leqq -1$	$x \geqq 1$ or $x \geqq -1$
Range of y	0 to π	0 to π	$-\frac{\pi}{2}$ to $\frac{\pi}{2}$
x positive	1st Quad.	1st Quad.	1st Quad.
x negative	2d Quad.	2d Quad.	4th Quad.

In so far as is possible we select the principal value of each inverse function, and its range, so that the function is single-valued, continuous, and takes on all possible values. This obviously cannot be done for the Arc sec x and for Arc csc y.

EXERCISES

1. Explain the difference between arc sin x and Arc sin x.

2. Find the values of the following expressions:

(*a*) Arc sin $\frac{1}{2}$.
(*b*) arc sin $\frac{1}{2}$.
(*c*) arc tan 1.
(*d*) Arc tan -1.
(*e*) arc cos $\frac{\sqrt{3}}{2}$.
(*f*) Arc cos $\frac{\sqrt{3}}{2}$.

3. What is meant by the angle π? $\pi/4$?

4. Through how many radians does the minute hand of a watch turn in 30 minutes? in one hour? in one and one half hours?

5. For what values of x are the following functions defined:

(*a*) arc sin x?
(*b*) arc cos x?
(*c*) arc tan x?
(*d*) arc ctn x?
(*e*) arc sec x?
(*f*) arc csc x?

6. What is the range of values of the functions:

(*a*) Arc sin x?
(*b*) Arc cos x?
(*c*) Arc tan x?
(*d*) Arc ctn x?
(*e*) Arc sec x.
(*f*) Arc csc x?

7. Draw the graph of the functions:

(*a*) arc sin x. (*c*) arc tan x. (*e*) arc sec x.
(*b*) arc cos x. (*d*) arc ctn x. (*f*) arc csc x.

8. Find the value of cos (Arc tan $\frac{3}{4}$).

HINT. Let Arc tan $\frac{3}{4} = \theta$. Then $\tan\theta = \frac{3}{4}$ and we wish to find the value of $\cos\theta$.

9. Find the values of cos (arc tan $\frac{3}{4}$).

10. Find the value of the following expressions:

(*a*) sin (arc cos $\frac{3}{5}$). (*c*) cos (Arc cos $\frac{5}{13}$). (*e*) sin (Arc sin $\frac{1}{5}$).
(*b*) sin (arc sec 3). (*d*) sec (Arc csc 2). (*f*) tan (Arc tan 5).

11. Prove that Arc sin $(2/5)=$ Arc tan $(2/\sqrt{21})$

12. Find x when Arc cos $(2x^2-2x)=2\pi/3$.

Find the values of the following expressions:

13. cos [90° − Arc tan $\frac{3}{4}$].

14. sec [90° − Arc sec 2].

15. tan [90° − Arc sin $\frac{5}{13}$].

67. Projection. Consider two directed lines p and q in a plane, *i.e.* two lines on each of which one of the directions has been specified as positive (Fig. 60). Let A and B be any two points on p and let A', B' be the points in which per-

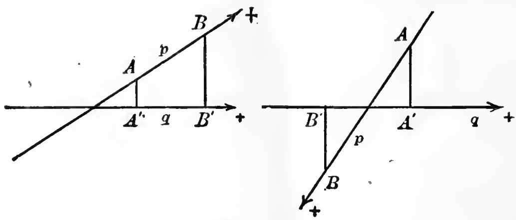


FIG. 60

pendiculars to q through A and B, respectively, meet q. The directed segment $A'B'$ is called the *projection of the directed segment AB on q* and is denoted by

$$A'B' = \text{proj}_q\, AB.$$

In both figures AB is positive. In the first figure $A'B'$ is positive, while in the second figure it is negative.

As special cases of this definition we note the following:

1. If p and q are parallel and are directed in the same way, we have

$$\text{proj}_q AB = AB.$$

2. If p and q are parallel and are directed oppositely, we have

$$\text{proj}_q AB = -AB.$$

3. If p is perpendicular to q, we have

$$\text{proj}_q AB = 0.$$

It should be noted carefully that these propositions are true no matter how A and B are situated on p.

We may now prove the following important proposition:

If A and B are any two points on a directed line p, and q is any directed line in the same plane with p, then we have both in magnitude and sign

$$(1) \qquad \text{proj}_q AB = AB \cdot \cos(pq)^* = AB \cdot \cos(qp).$$

We note first from § 8 that $(pq) + (qp) = 0 + n \cdot 360°$, where n is any integer. Hence from § 64, $\cos(pq) = \cos(qp)$. Two cases arise.

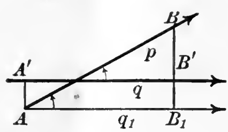


FIG. 61

CASE 1. Suppose AB is positive, *i.e.* it has the same direction as p.

Through A draw a line q_1 parallel to q and with the same direction. [It is evident that we may assume without loss of generality that q is horizontal and is directed to the right.] Let $A'B'$ be the projection of AB on q and let BB' meet q_1 in B_1. Then by the definition of the cosine we have

$$\frac{AB_1}{AB} = \cos(q_1p) = \cos(pq_1) = \cos(qp) = \cos(pq)$$

* (pq) represents an angle through which p may be rotated in order to make its direction coincide with the direction of q; similarly for (qp).

in magnitude and sign. Hence

$$AB_1 = AB \cdot \cos(pq) = AB \cdot \cos(qp).$$

But $\quad AB_1 = A'B' = \text{proj}_q AB.$

Therefore $\quad \text{proj}_q AB = AB \cdot \cos(pp) = AB \cdot \cos(qp).$

CASE 2. Suppose AB is negative.

If AB is negative, BA is positive and we have from Case 1,

$$B'A' = BA \cdot \cos(pq) = BA \cdot \cos(qp).$$

Changing the signs of both members of this equation, we have

$$A'B' = AB \cdot \cos(pq) = AB \cdot \cos(qp).$$

The special cases 1, 2, 3, are obtained from formula (**1**) by placing (qp) or (pq) equal to 0°, 180°, 90° respectively.

THEOREM. *If A, B, C are any three points in a plane, and l is any directed line in the plane, the algebraic sum of the projections of the segments AB and BC on l is equal to the projection of the segment AC on l.*

As a point traces out the path from A to B, and then from B to C (Fig. 62), the projection of the point traces out the segments from A' to B' and then from B' to C'. The net result of this motion is a motion from A' to C' which represents the projection of AC, *i.e.*

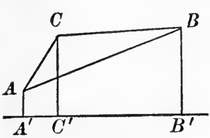


FIG. 62

$$A'B' + B'C' = A'C'.$$

EXERCISES

1. What is the projection of a line segment upon a line l, if the line segment is perpendicular to the line l?

2. Find $\text{proj}_x AB$ and $\text{proj}_y AB$* in each of the following cases, if α denotes the angle from the x-axis to AB.

(*a*) $AB = 5$,	$\alpha = 60°$.	(*c*) $AB = 6$,	$\alpha = 90°$.
(*b*) $AB = 10$,	$\alpha = 300°$.	(*d*) $AB = 20$,	$\alpha = 210°$.

* $\text{Proj}_x AB$ and $\text{proj}_y AB$ mean the projections of AB on the x-axis and the y-axis, respectively.

3. Prove by means of projection that in a triangle ABC

$$a = b \cos C + c \cos B.$$

4. If $\text{proj}_x AB = 3$ and $\text{proj}_y AB = -4$, find the length of AB.

5. A steamer is going northeast 20 miles per hour. How fast is it going north? going east?

6. A 20 lb. block is sliding down a 15° incline. Find what force acting directly up the plane will just hold the block, allowing one half a pound for friction.

7. Prove that if the sides of a polygon are projected in order upon any given line, the sum of these projections is zero.

68. The Addition Formulas. We may now derive formulas for $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, and $\tan(\alpha+\beta)$ in terms of functions of α and β. To this end let $P(x, y)$ be any point on the terminal side of the angle α (the initial side being along the positive end of the x-axis and the vertex being at the origin). The angle $\alpha+\beta$ is then obtained by rotating OP through an angle β. If $P'(x', y')$ is the new position P after this rotation and

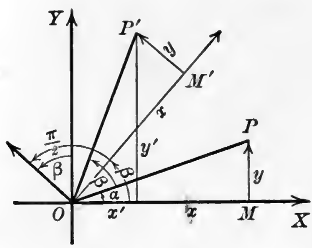


FIG. 63

$OP = OP' = r$, we have $\sin(\alpha+\beta) = \dfrac{y'}{r}$, $\cos(\alpha+\beta) = \dfrac{x'}{r}$, by definition. Our first problem is, therefore, to find x' and y' in terms of x, y, and β.

In the figure $OM'P'$ is the new position of the triangle OMP after rotating it about O through the angle β. Now,

$$\begin{aligned} x' = \text{proj}_x OP' &= \text{proj}_x OM' + \text{proj}_x M'P' \\ &= x\cos\beta + y\cos\left(\beta+\frac{\pi}{2}\right) \\ &= x\cos\beta - y\sin\beta. \end{aligned}$$

Similarly,

$$\begin{aligned} y' = \text{proj}_y\, OP' &= \text{proj}_y\, OM' + \text{proj}_y\, M'P' \\ &= x \cos\left(\frac{\pi}{2} - \beta\right) + y \cos\beta \\ &= x \sin\beta + y\cos\beta. \end{aligned}$$

Hence, $\sin(\alpha + \beta) = \frac{y'}{r} = \frac{x}{r}\sin\beta + \frac{y}{r}\cos\beta$

or (1) $\mathbf{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.}$

Also $\cos(\alpha + \beta) = \frac{x'}{r} = \frac{x}{r}\cos\beta - \frac{y}{r}\sin\beta.$

or (2) $\mathbf{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.}$

Further we have

$$\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing numerator and denominator by $\cos\alpha\cos\beta$, we have

(3) $$\mathbf{\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.}$$

Furthermore, by replacing β by $-\beta$ in (1), (2), and (3), and recalling that

$\sin(-\beta) = -\sin\beta$, $\cos(-\beta) = \cos\beta$, $\tan(-\beta) = -\tan\beta$,

we obtain

(4) $\mathbf{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,}$

(5) $\mathbf{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,}$

(6) $$\mathbf{\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.}$$

EXERCISES

Expand the following:

1. $\sin(45° + \alpha) =$ **3.** $\cos(60° + \alpha) =$ **5.** $\sin(30° - 45°) =$

2. $\tan(30° - \beta) =$ **4.** $\tan(45° + 60°) =$ **6.** $\cos(180° - 45°) =$

7. What do the following formulas become if $\alpha = \beta$?

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$

$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

8. Complete the following formulas:

$\sin 2\alpha \cos \alpha + \cos 2\alpha \sin \alpha =$ $\qquad \dfrac{\tan 2\alpha + \tan \alpha}{1 - \tan 2\alpha \tan \alpha} =$

$\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha =$

9. Prove $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$, $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

10. Given $\tan \alpha = \frac{3}{4}$, $\sin \beta = \frac{5}{13}$, and α and β both positive acute angles, find the value of $\tan(\alpha+\beta)$; $\sin(\alpha-\beta)$; $\cos(\alpha+\beta)$; $\tan(\alpha-\beta)$.

11. Prove that

(*a*) $\cos(60° + \alpha) + \sin(30° + \alpha) = \cos \alpha$.

(*b*) $\sin(60° + \theta) - \sin(60° - \theta) = \sin \theta$.

(*c*) $\cos(30° + \theta) - \cos(30° - \theta) = -\sin \theta$.

(*d*) $\cos(45° + \theta) + \cos(45° - \theta) = \sqrt{2} \cdot \cos \theta$.

(*e*) $\sin\left(\alpha + \dfrac{\pi}{3}\right) + \sin\left(\alpha - \dfrac{\pi}{3}\right) = \sin \alpha$.

(*f*) $\cos\left(\alpha + \dfrac{\pi}{6}\right) + \cos\left(\alpha - \dfrac{\pi}{6}\right) = \sqrt{3} \cdot \cos \alpha$.

(*g*) $\tan(45° + \theta) = \dfrac{1 + \tan \theta}{1 - \tan \theta}$. $\qquad$ (*h*) $\tan(45° - \theta) = \dfrac{1 - \tan \theta}{1 + \tan \theta}$.

12. By using the functions of 60° and 30° find the value of sin 90°; cos 90°.

13. Find in radical form the value of sin 15°; cos 15°; tan 15°; sin 105°; cos 105°; tan 105°.

14. If $\tan \alpha = \frac{4}{3}$, $\sin \beta = \frac{5}{13}$, and α is in the third quadrant while β is in the second, find $\sin(\alpha \pm \beta)$; $\cos(\alpha \pm \beta)$; $\tan(\alpha \pm \beta)$.

Prove the following identities:

15. $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta}$. $\qquad$ **16.** $\dfrac{\sin 2\alpha}{\sec \alpha} + \dfrac{\cos 2\alpha}{\csc \alpha} = \sin 3\alpha$.

17. $\dfrac{\tan \alpha - \tan(\alpha-\beta)}{1 + \tan \alpha \tan(\alpha-\beta)} = \tan \beta$.

18. $\tan(\theta \pm 45°) + \text{ctn}(\theta \mp 45°) = 0$.

19. (*a*) $\sin(180° - \theta) = \sin \theta$.

(*b*) $\cos(180° - \theta) = -\cos \theta$.

(*c*) $\tan(180° - \theta) = -\tan \theta$.

20. $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2 \alpha - \sin^2 \beta$.

21. $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2 \alpha - \sin^2 \beta$.

22. $\text{ctn}(\alpha+\beta) = \dfrac{\text{ctn}\,\alpha\,\text{ctn}\,\beta - 1}{\text{ctn}\,\alpha + \text{ctn}\,\beta}$. $\qquad$ **23.** $\text{ctn}(\alpha-\beta) = \dfrac{\text{ctn}\,\alpha\,\text{ctn}\,\beta + 1}{\text{ctn}\,\beta - \text{ctn}\,\alpha}$.

24. Prove $\text{Arc tan}\,\frac{1}{2} + \text{Arc tan}\,\frac{1}{3} = \pi/4$.

[Hint: Let $\text{Arc tan}\,\frac{1}{2} = x$ and $\text{Arc tan}\,\frac{1}{3} = y$. Then we wish to prove $x + y = \pi/4$, which is true since $\tan(x+y) = 1$.]

25. Prove $\text{Arc sin}\,a + \text{Arc cos}\,a = \dfrac{\pi}{2}$ if $0 < a < 1$.

26. Prove $\text{Arc sin}\,\frac{8}{17} + \text{Arc sin}\,\frac{3}{5} = \text{Arc sin}\,\frac{77}{85}$.

27. Prove Arc tan $2+$ Arc tan $\frac{1}{2}=\pi/2$.

28. Prove Arc cos $\frac{3}{5}+$ Arc cos $(-\frac{5}{13})=$ Arc cos $(-\frac{63}{65})$.

29. Prove Arc tan $\frac{8}{15}+$ Arc tan $\frac{3}{4}=$ Arc tan $\frac{77}{36}$.

30. Find the value of sin [Arc sin $\frac{4}{5}+$ Arc ctn $\frac{4}{3}$].

31. Find the value of sin [Arc sin $a+$ Arc sin b] if $0<a<1, 0<b<1$.

32. Expand $\sin(x+y+z)$; $\cos(x+y+z)$.

[HINT: $x+y+z=(x+y)+z$.]

33. The area A of a triangle was computed from the formula $A=\frac{1}{2}ab\sin\theta$. If an error ϵ was made in measuring the angle θ, show that the corrected area A' is given by the relation $A'=A(\cos\epsilon+\sin\epsilon\operatorname{ctn}\theta)$.

69. Functions of Double Angles. In this and the following articles (§§ 69–71) we shall derive from the addition formulas a variety of other relations which are serviceable in transforming trigonometric expressions. Since the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ are true for all angles α and β, they will be true when $\beta=\alpha$. Putting $\beta=\alpha$, we obtain

(1) $$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$

(2) $$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$

Since $\sin^2\alpha+\cos^2\alpha=1$, we have also

(3) $$\cos 2\alpha = 1 - 2\sin^2\alpha$$

(4) $$= 2\cos^2\alpha - 1.$$

Similarly the formula for $\tan(\alpha+\beta)$ (which is true for all angles α, β, and $\alpha+\beta$ which have tangents) becomes, when $\beta=\alpha$,

(5) $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

which holds for every angle for which both members are defined.

The above formulas should be learned in words. For example, formula (1) states that the sine of any angle equals twice the sine of half the angle times the cosine of half the angle. Thus

$$\sin 6x = 2\sin 3x\cos 3x,$$

$$\tan 4x = \frac{2\tan 2x}{1-\tan^2 2x},$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}.$$

70. Functions of Half Angles. From (3), § 69, we have

$$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha.$$

Therefore

(6) $$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

From (4), § 69, we have

$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha.$$

Therefore

(7) $$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$

Formulas (6) and (7) are at once seen to hold for all angles α. Now, if we divide formula (6) by formula (7), we obtain

(8) $$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},$$

which is true for all angles α except $n \cdot 180°$, where n is any odd integer.

Example. Given $\sin A = -3/5$, $\cos A$ negative; find $\sin(A/2)$.

Since the angle A is in the third quadrant, $A/2$ is in the second or fourth quadrant, and hence $\sin(A/2)$ may be either positive or negative. Therefore, since $\cos A = -4/5$, we have

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1+\frac{4}{5}}{2}} = \pm\frac{3}{\sqrt{10}} = \pm\frac{3}{10}\sqrt{10}.$$

EXERCISES

Complete the following formulas and state whether they are true for all angles:

1. $\sin 2\alpha =$
2. $\cos 2\alpha =$ (three forms).
3. $\tan 2\alpha =$
4. $\sin\frac{\alpha}{2} =$
5. $\cos\frac{\alpha}{2} =$
6. $\tan\frac{\alpha}{2} =$
7. In what quadrant is $\theta/2$ if θ is positive, less than 360°, and in the second quadrant? third quadrant? fourth quadrant?
8. Express $\cos 2\alpha$ in terms of $\cos 4\alpha$.
9. Express $\sin 6x$ in terms of functions of $3x$.

10. Express $\tan 4\alpha$ in terms of $\tan 2\alpha$.

11. Express $\tan 4\alpha$ in terms of $\cos 8\alpha$.

12. Express $\sin x$ in terms of functions of $x/2$.

13. Explain why the formulas for $\sin x$ and $\cos x$ in terms of functions of $2x$ have a double sign.

14. From the functions of 30° find those of 60°.

15. From the functions of 60° find those of 30°.

16. From the functions of 30° find those of 15°.

17. From the functions of 15° find those of 7°.5.

18. Find the functions of 2α if $\sin\alpha = \frac{3}{5}$ and α is in the second quadrant.

19. Find the functions of $\alpha/2$ if $\cos\alpha = -0.6$ and α is in the third quadrant, positive, and less than 360°.

20. Express $\sin 3\alpha$ in terms of $\sin\alpha$. [HINT: $3\alpha = 2\alpha + \alpha$.]

21. From the value of cos 45° find the functions of 22°.5.

22. Given $\sin\alpha = \dfrac{5}{13}$ and α in the second quadrant. Find the values of

(*a*) $\sin 2a$. (*c*) $\cos 2\alpha$. (*e*) $\tan 2\alpha$.

(*b*) $\sin\dfrac{\alpha}{2}$. (*d*) $\cos\dfrac{\alpha}{2}$. (*f*) $\tan\dfrac{\alpha}{2}$.

23. If $\tan 2\alpha = \frac{3}{4}$ find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ if α is an angle in the third quadrant.

Prove the following identities:

24. $\dfrac{1+\cos\alpha}{\sin\alpha} = \text{ctn}\,\dfrac{\alpha}{2}$.

25. $\left[\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right]^2 = 1 - \sin\theta$.

26. $\dfrac{\cos 2\theta + \cos\theta + 1}{\sin 2\theta + \sin\theta} = \text{ctn}\,\theta$.

27. $\dfrac{1-\cos 2\theta + \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \tan\theta$.

28. $\sin\dfrac{\alpha}{2} + \cos\dfrac{\alpha}{2} = \pm\sqrt{1+\sin\alpha}$.

29. $\sec\alpha + \tan\alpha = \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$.

30. $2\,\text{Arc}\cos x = \text{Arc}\cos(2x^2-1)$.

31. $2\,\text{Arc}\cos x = \text{Arc}\sin(2x\sqrt{1-x^2})$.

32. $\tan[2\,\text{Arc}\tan x] = \dfrac{2x}{1-x^2}$.

33. $\cos[2\,\text{Arc}\tan x] = \dfrac{1-x^2}{1+x^2}$.

34. $\tan[2\,\text{Arc}\sec x] = \pm\dfrac{2\sqrt{x^2-1}}{2-x^2}$.

35. $\cos(2\,\text{Arc}\sin a) = 1 - 2a^2$.

Solve the following equations:

36. $\cos 2x + 5\sin x = 3$.

37. $\cos 2x - \sin x = \frac{1}{2}$.

38. $\sin 2x\cos x = \sin x$.

39. $2\sin^2 x + \sin^2 2x = 2$.

40. $\sin^2 2x - \sin^2 x = \frac{3}{4}$.

41. $\sin 2x = 2\cos x$.

42. $2\sin^2 2x = 1 - \cos 2x$.

43. $\text{ctn}\,x - \csc 2x - 1$.

44. A flagpole 50 ft. high stands on a tower 49 ft. high. At what distance from the foot of the tower will the flagpole and the tower subtend equal angles?

45. The dial of a town clock has a diameter of 10 ft. and its center is 100 ft. above the ground. At what distance from the foot of the tower will the dial be most plainly visible? [The angle subtended by the dial must be as large as possible.]

71. Product Formulas. From § 68 we have

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Adding, we get

(1) $$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta.$$

Subtracting, we have

(2) $$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta.$$

Now, if we let $\alpha + \beta = P$ and $\alpha - \beta = Q$,

then $$\alpha = \frac{P+Q}{2}, \quad \beta = \frac{P-Q}{2}.$$

Therefore formulas (1) and (2) become

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2},$$
$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

Similarly, starting with $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$ and performing the same operations, the following formulas result:

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2},$$
$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

In words:

the sum of two sines =
twice sin (half sum) times cos (half difference),

the difference of two sines =
twice cos (half sum) times sin (half difference),*

* The difference is taken, first angle minus the second.

the sum of two cosines =

twice cos (half sum) times cos (half difference),

the difference of two cosines =

minus twice sin (half sum) times sin (half difference).*

EXAMPLE 1. Prove that

$$\frac{\cos 3x + \cos x}{\sin 3x + \sin x} = \operatorname{ctn} 2x,$$

for all angles for which both members are defined.

$$\frac{\cos 3x + \cos x}{\sin 3x + \sin x} = \frac{2\cos\frac{1}{2}(3x + x)\cos\frac{1}{2}(3x - x)}{2\sin\frac{1}{2}(3x + x)\cos\frac{1}{2}(3x - x)} = \frac{\cos 2x}{\sin 2x} = \operatorname{ctn} 2x.$$

EXAMPLE 2. Reduce $\sin 4x + \cos 2x$ to the form of a product.

We may write this as $\sin 4x + \sin(90° - 2x)$, which is equal to

$$2\sin\frac{4x + 90° - 2x}{2}\cos\frac{4x - 90° + 2x}{2} = 2\sin(45° + x)\cos(3x - 45°).$$

EXERCISES

Reduce to a product:

1. $\sin 4\theta - \sin 2\theta$.

2. $\cos\theta + \cos 3\theta$.

3. $\cos 6\theta + \cos 2\theta$.

4. $\cos 2\theta + \sin 2\theta$.

5. $\cos 3\theta - \cos 6\theta$.

6. $\sin(x + \Delta x) - \sin x$.

7. $\cos 3x + \sin 5x$.

8. $\sin 20° - \sin 60°$.

Show that

9. $\sin 20° + \sin 40° = \cos 10°$.

10. $\cos 50° + \cos 70° = \cos 10°$.

11. $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \tan 30°$.

12. $\dfrac{\sin 15° + \sin 75°}{\sin 15° - \sin 75°} = -\tan 60°$.

13. $\dfrac{\sin 3\theta - \sin 5\theta}{\cos 3\theta - \cos 5\theta} = -\operatorname{ctn} 4\theta$.

Prove the following identities:

14. $\dfrac{\sin 4\alpha + \sin 3\alpha}{\cos 3\alpha - \cos 4\alpha} = \operatorname{ctn}\dfrac{\alpha}{2}$.

15. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \dfrac{\tan\frac{1}{2}(\alpha + \beta)}{\tan\frac{1}{2}(\alpha - \beta)}$.

16. $\dfrac{\cos\alpha + 2\cos 3\alpha + \cos 5\alpha}{\cos 3\alpha + 2\cos 5\alpha + \cos 7\alpha} = \dfrac{\cos 3\alpha}{\cos 5\alpha}$.

17. $\dfrac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta} = -\dfrac{\tan\frac{1}{2}(\alpha + \beta)}{\operatorname{ctn}\frac{1}{2}(\alpha - \beta)}$.

18. $\dfrac{\sin(n - 2)\theta + \sin n\theta}{\cos(n - 2)\theta - \cos n\theta} = \operatorname{ctn}\theta$.

Solve the following equations:

19. $\cos\theta + \cos 5\theta = \cos 3\theta$.

20. $\sin\theta + \sin 5\theta = \sin 3\theta$.

21. $\sin 3\theta + \sin 7\theta = \sin 5\theta$.

22. $\sin 4\theta - \sin 2\theta = \cos 3\theta$.

23. $\cos 7\theta - \cos\theta = -\sin 4\theta$.

* The difference is taken, first angle minus the second.

MISCELLANEOUS EXERCISES

1. Reduce to radians 65°, − 135°, − 300°, 20°.

2. Reduce to degrees π, 3π, -2π, 4π radians.

3. Find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$ when it is given that α and β are positive and acute and $\tan\alpha = \frac{3}{4}$ and $\sec\beta = \frac{13}{5}$.

4. Find $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$ when it is given that $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$.

5. Prove that $\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$.

6. Given $\sin\theta = \dfrac{2}{\sqrt{5}}$, and θ in the second quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

Prove the following identities:

7. $\sin 2\alpha = \dfrac{2\tan\alpha}{1 + \tan^2\alpha}$.

8. $\cos 2\alpha = \dfrac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$.

9. $\sec 2\alpha = \dfrac{\csc^2\alpha}{\csc^2\alpha - 2}$.

10. $\tan\alpha = \dfrac{\sin 2\alpha}{1 + \cos 2\alpha}$.

11. $\sin(\alpha + \beta)\cos\beta - \cos(\alpha + \beta)\sin\beta = \sin\alpha$.

12. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$, if $\alpha + \beta + \gamma = 180°$.

13. $\dfrac{\cos\alpha}{1 - \sin\alpha} = \dfrac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}$.

$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$

$2\cos\frac{1}{2}(\theta + 5\theta)\cos\frac{1}{2}(\theta - 5\theta) = \cos 3$

$2\cos 3\theta\cos 2\theta = \cos 3\theta$

$-2\cos 2\theta = 1$

$\cos 2\theta = -\frac{1}{2}$

$\cos^2\theta - \sin^2\theta = -\frac{1}{2}$

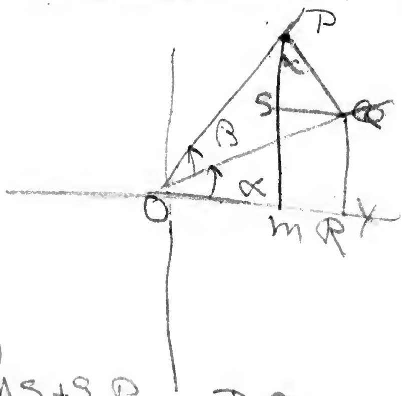
$1 - 2\sin^2\theta = -\frac{1}{2}$

$2\sin^2\theta = \frac{3}{2}$

$\sin^2 = \frac{3}{4}$

$\sin = \frac{\sqrt{3}}{2}$

Section 68.



Sin α + β

$$\sin(\alpha+\beta) = \frac{MP}{OP}$$

$$\sin(\alpha+\beta) = \frac{MS+SP}{OP} = \frac{RQ}{OP} + \frac{SP}{OP}$$

$$= \frac{RQ}{OQ}\,\frac{OQ}{OP} + \frac{SP}{PQ}\,\frac{PQ}{OP}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

2π Radians = 360°

π . R = 180°

$\frac{\pi}{2}$ R = 90°

$\frac{\pi}{4}$ R = 45°

$\frac{\pi}{6}$ R = 30°

$\frac{\text{arc}}{\text{Rad}} = \angle^R$

arc = Rad × $\angle^R$

TABLES

$180^\circ = \pi^R$

TO

FOUR DECIMAL PLACES

$\frac{2\pi R}{R}$ linear units = 2π Radians

Squares of Numbers

[Moving the decimal point *one* place in N requires a corresponding move of *two* places in N^2]

N	N^2 0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0001	.0004	.0009	.0016	.0025	.0036	.0049	.0064	.0081
0.1	.0100	.0121	.0144	.0169	.0196	.0225	.0256	.0289	.0324	.0361
0.2	.0400	.0441	.0484	.0529	.0576	.0625	.0676	.0729	.0784	.0841
0.3	.0900	.0961	.1024	.1089	.1156	.1225	.1296	.1369	.1444	.1521
0.4	.1600	.1681	.1764	.1849	.1936	.2025	.2116	.2209	.2304	.2401
0.5	.2500	.2601	.2704	.2809	.2916	.3025	.3136	.3249	.3364	.3481
0.6	.3600	.3721	.3844	.3969	.4096	.4225	.4356	.4489	.4624	.4761
0.7	.4900	.5041	.5184	.5329	.5476	.5625	.5776	.5929	.6084	.6241
0.8	.6400	.6561	.6724	.6889	.7056	.7225	.7396	.7569	.7744	.7921
0.9	.8100	.8281	.8464	.8649	.8836	.9025	.9216	.9409	.9604	.9801
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.652	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.573	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.39	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.26
3.8	14.44	14.52	14.59	14.70	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.65	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91

Squares of Numbers

[Moving the decimal point *one* place in N requires a corresponding move of *two* places in N^2]

N	N^2 0	1	2	3	4	5	6	7	8	9
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.42	45.56	45.70	45.83	45.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.72
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.08	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.00	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.99	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

SQUARES AND CUBES   SQUARE ROOTS AND CUBE ROOTS

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	**51**	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	**52**	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	**53**	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	**54**	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	**55**	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	**56**	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	**57**	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	**58**	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	**59**	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	**61**	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	**62**	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	**63**	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	**64**	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	**65**	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	**66**	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	**67**	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	**68**	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	**69**	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	**71**	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	**72**	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	**73**	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	**74**	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	**76**	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	**77**	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	**78**	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	**79**	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	**81**	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	**82**	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	**83**	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	**84**	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	**85**	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	**86**	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	**87**	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	**88**	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	**89**	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	**90**	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	**91**	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	**92**	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	**93**	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	**94**	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	**95**	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	**96**	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	**97**	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	**98**	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	**99**	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.642

For a more complete table, see THE MACMILLAN TABLES, pp. 94–111.

CERTAIN CONVENIENT VALUES FOR $n = 1$ TO $n = 10$

n	$1/n$	$\sqrt{n}$	$\sqrt[3]{n}$	$n!$	$1/n!$	$\log_{10} n$
1	1.000000	1.00000	1.00000	1	1.0000000	0.000000000
2	0.500000	1.41421	1.25992	2	0.5000000	0.301029996
3	0.333333	1.73205	1.44225	6	0.1666667	0.477121255
4	0.250000	2.00000	1.58740	24	0.0416667	0.602059991
5	0.200000	2.23607	1.70998	120	0.0083333	0.698970004
6	0.166667	2.44949	1.81712	720	0.0013889	0.778151250
7	0.142857	2.64575	1.91293	5040	0.0001984	0.845098040
8	0.125000	2.82843	2.00000	40320	0.0000248	0.903089987
9	0.111111	3.00000	2.08008	362880	0.0000028	0.954242509
10	0.100000	3.16228	2.15443	3628800	0.0000003	1.000000000

LOGARITHMS OF IMPORTANT CONSTANTS

n = NUMBER	VALUE OF n	$\log_{10} n$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Napierian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812262
$\pi \div 180$ = radians in 1°	0.01745329	8.24187738
$\pi \div 10800$ = radians in 1′	0.0002908882	6.46372613
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811152	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37 (exact legal value)	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g (average value)	32.16 ft./sec./sec.	1.5073
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953586
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.907
cu. in. in 1 (U. S.) gallon	231 (exact legal value)	2.3636120
ft. lb. per sec. in 1 H. P.	550. (exact legal value)	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1.8810445
watts in 1 H. P.	745.957	2.8727135

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9	10	12	14	16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	0	1	2	3	4	5	6	7	8	9	1	2	2	4	5	6	7	8	9

The proportional parts are stated in full for every tenth at the right-hand side. The logarithm of any number of four significant figures can be read directly by add-

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ing the proportional part corresponding to the fourth figure to the tabular number corresponding to the first three figures. There may be an error of 1 in the last place.

Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine		Tangent		Cotangent		Cosine			
		Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Value	Log_{10}		
.0000	**0° 00′**	.0000	—	.0000	—	—	—	1.0000	.0000	**90° 00′**	1.5708
.0029	10	.0029	.4637	.0029	.4637	343.77	.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	.9408	.0087	.9409	114.59	.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	.0658	.0116	.0658	85.940	.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	.0000	10	1.5563
.0175	**1° 00′**	.0175	.2419	.0175	.2419	57.290	.7581	.9998	.9999	**89° 00′**	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	**2° 00′**	.0349	.5428	.0349	.5431	28.636	.4569	.9994	.9997	**88° 00′**	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	.9995	10	1.5213
.0524	**3° 00′**	.0523	.7188	.0524	.7194	19.081	.2806	.9986	.9994	**87° 00′**	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	**4° 00′**	.0698	.8436	.0699	.8446	14.301	.1554	.9976	.9989	**86° 00′**	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	**5° 00′**	.0872	.9403	.0875	.9420	11.430	.0580	.9962	.9983	**85° 00′**	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	.9945	.0992	.9966	10.078	.0034	.9951	.9979	20	1.4719
.1018	50	.1016	.0070	.1022	.0093	9.7882	.9907	.9948	.9977	10	1.4690
.1047	**6° 00′**	.1045	.0192	.1051	.0216	9.5144	.9784	.9945	.9976	**84° 00′**	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	**7° 00′**	.1219	.0859	.1228	.0891	8.1443	.9109	.9925	.9968	**83° 00′**	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	**8° 00′**	.1392	.1436	.1405	.1478	7.1154	.8522	.9903	.9958	**82° 00′**	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	**9° 00′**	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	**81° 00′**	1.4137
		Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine Log_{10}	Tangent Value	Tangent Log_{10}	Cotangent Value	Cotangent Log_{10}	Cosine Value	Cosine Log_{10}		
.1571	**9° 00′**	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	**81° 00′**	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
.1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	**10° 00′**	.1736	.2397	.1763	.2463	5.6713	.7537	.9848	.9934	**80° 00′**	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	**11° 00′**	.1908	.2806	.1944	.2887	5.1446	.7113	.9816	.9919	**79° 00′**	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	**12° 00′**	.2079	.3179	.2126	.3275	4.7046	.6725	.9781	.9904	**78° 00′**	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	**13° 00′**	.2250	.3521	.2309	.3634	4.3315	.6366	.9744	.9887	**77° 00′**	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	**14° 00′**	.2419	.3837	.2493	.3968	4.0108	.6032	.9703	.9869	**76° 00′**	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	**15° 00′**	.2588	.4130	.2679	.4281	3.7321	.5719	.9659	.9849	**75° 00′**	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	**16° 00′**	.2756	.4403	.2867	.4575	3.4874	.5425	.9613	.9828	**74° 00′**	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	**17° 00′**	.2924	.4659	.3057	.4853	3.2709	.5147	.9563	.9806	**73° 00′**	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	.9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	**18° 00′**	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	**72° 00′**	1.2566
		Value Cosine	Log_{10}	Value Cotangent	Log_{10}	Value Tangent	Log_{10}	Value Sine	Log_{10}	Degrees	Radians

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine Log_{10}	Tangent Value	Tangent Log_{10}	Cotangent Value	Cotangent Log_{10}	Cosine Value	Cosine Log_{10}		
.3142	**18° 00′**	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	**72° 00′**	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	**19° 00′**	.3256	.5126	.3443	.5370	2.9042	.4630	.9455	.9757	**71° 00′**	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	**20° 00′**	.3420	.5341	.3640	.5611	2.7475	.4389	.9397	.9730	**70° 00′**	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	**21° 00′**	.3584	.5543	.3839	.5842	2.6051	.4158	.9336	.9702	**69° 00′**	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	**22° 00′**	.3746	.5736	.4040	.6064	2.4751	.3936	.9272	.9672	**68° 00′**	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	.9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	**23° 00′**	.3907	.5919	.4245	.6279	2.3559	.3721	.9205	.9640	**67° 00′**	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	**24° 00′**	.4067	.6093	.4452	.6486	2.2460	.3514	.9135	.9607	**66° 00′**	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	**25° 00′**	.4226	.6259	.4663	.6687	2.1445	.3313	.9063	.9573	**65° 00′**	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	.9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	**26° 00′**	.4384	.6418	.4877	.6882	2.0503	.3118	.8988	.9537	**64° 00′**	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	.3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	.3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	**27° 00′**	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	**63° 00′**	1.0996
		Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine Log_{10}	Tangent Value	Tangent Log_{10}	Cotangent Value	Cotangent Log_{10}	Cosine Value	Cosine Log_{10}		
.4712	**27° 00′**	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	**63° 00′**	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	**28° 00′**	.4695	.6716	.5317	.7257	1.8807	.2743	.8829	.9459	**62° 00′**	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5467	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	**29° 00′**	.4848	.6856	.5543	.7438	1.8040	.2562	.8746	.9418	**61° 00′**	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	.6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	**30° 00′**	.5000	.6990	.5774	.7614	1.7321	.2386	.8660	.9375	**60° 00′**	1.0472
.5265	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	.7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	**31° 00′**	.5150	.7118	.6009	.7788	1.6643	.2212	.8572	.9331	**59° 00′**	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	**32° 00′**	.5299	.7242	.6249	.7958	1.6003	.2042	.8480	.9284	**58° 00′**	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	**33° 00′**	.5446	.7361	.6494	.8125	1.5399	.1875	.8387	.9236	**57° 00′**	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	**34° 00′**	.5592	.7476	.6745	.8290	1.4826	.1710	.8290	.9186	**56° 00′**	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	**35° 00′**	.5736	.7586	.7002	.8452	1.4281	.1548	.8192	.9134	**55° 00′**	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	**36° 00′**	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	**54° 00′**	.9425
		Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Value	Log_{10}	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine Log_{10}	Tangent Value	Tangent Log_{10}	Cotangent Value	Cotangent Log_{10}	Cosine Value	Cosine Log_{10}		
.6283	**36° 00′**	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	**54° 00′**	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	**37° 00′**	.6018	.7795	.7536	.8771	1.3270	.1229	.7986	.9023	**53° 00′**	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.7934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	**38° 00′**	.6157	.7893	.7813	.8928	1.2799	.1072	.7880	.8965	**52° 00′**	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	**39° 00′**	.6293	.7989	.8098	.9084	1.2349	.0916	.7771	.8905	**51° 00′**	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	**40° 00′**	.6428	.8081	.8391	.9238	1.1918	.0762	.7660	.8843	**50° 00′**	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	**41° 00′**	.6561	.8169	.8693	.9392	1.1504	.0608	.7547	.8778	**49° 00′**	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	**42° 00′**	.6691	.8255	.9004	.9544	1.1106	.0456	.7431	.8711	**48° 00′**	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	**43° 00′**	.6820	.8338	.9325	.9697	1.0724	.0303	.7314	.8641	**47° 00′**	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	**44° 00′**	.6947	.8418	.9657	.9848	1.0355	.0152	.7193	.8569	**46° 00′**	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	.9975	1.0058	.0025	.7092	.8507	10	.7883
.7854	**45° 00′**	.7071	.8495	1.0000	.0000	1.0000	.0000	.7071	.8495	**45° 00′**	.7854
		Value Cosine	Log_{10}	Value Cotangent	Log_{10}	Value Tangent	Log_{10}	Value Sine	Log_{10}	Degrees	Radians

[Characteristics of Logarithms omitted — determine by rule from the value]

°	0′ Value	0′ Log_{10}	10′ Value	10′ Log_{10}	20′ Value	20′ Log_{10}	30′ Value	30′ Log_{10}	40′ Value	40′ Log_{10}	50′ Value	50′ Log_{10}
0	.0000		.0000	4.3254	.0000	4.9275	.0000	5.2796	.0000	5.5295	.0001	5.7233
1	.0001	5.8817	.0001	6.0156	.0001	6.1315	.0002	.2338	.0002	.3254	.0003	.4081
2	.0003	.4837	.0004	.5532	.0004	.6176	.0005	.6775	.0005	.7336	.0006	.7862
3	.0007	.8358	.0008	.8828	.0008	.9273	.0009	.9697	.0010	.0101	.0011	.0487
4	.0012	.0856	.0013	.1211	.0014	.1551	.0015	.1879	.0017	.2195	.0018	.2499
5	.0019	.2793	.0020	.3078	.0022	.3354	.0023	.3621	.0024	.3880	.0026	.4132
6	.0027	.4376	.0029	.4614	.0031	.4845	.0032	.5071	.0034	.5290	.0036	.5504
7	.0037	.5713	.0039	.5918	.0041	.6117	.0043	.6312	.0045	.6503	.0047	.6689
8	.0049	.6872	.0051	.7051	.0053	.7226	.0055	.7397	.0057	.7566	.0059	.7731
9	.0062	.7893	.0064	.8052	.0066	.8208	.0069	.8361	.0071	.8512	.0073	.8660
10	.0076	.8806	.0079	.8949	.0081	.9090	.0084	.9229	.0086	.9365	.0089	.9499
11	.0092	.9631	.0095	.9762	.0097	.9890	.0100	.0016	.0103	.0141	.0106	.0264
12	.0109	.0385	.0112	.0504	.0115	.0622	.0119	.0738	.0122	.0853	.0125	.0966
13	.0128	.1077	.0131	.1187	.0135	.1296	.0138	.1404	.0142	.1510	.0145	.1614
14	.0149	.1718	.0152	.1820	.0156	.1921	.0159	.2021	.0163	.2120	.0167	.2218
15	.0170	.2314	.0174	.2409	.0178	.2504	.0182	.2597	.0186	.2689	.0190	.2781
16	.0194	.2871	.0198	.2961	.0202	.3049	.0206	.3137	.0210	.3223	.0214	.3309
17	.0218	.3394	.0223	.3478	.0227	.3561	.0231	.3644	.0236	.3726	.0240	.3806
18	.0245	.3887	.0249	.3966	.0254	.4045	.0258	.4123	.0263	.4200	.0268	.4276
19	.0272	.4352	.0277	.4427	.0282	.4502	.0287	.4576	.0292	.4649	.0297	.4721
20	.0302	.4793	.0307	.4865	.0312	.4936	.0317	.5006	.0322	.5075	.0327	.5144
21	.0332	.5213	.0337	.5281	.0343	.5348	.0348	.5415	.0353	.5481	.0359	.5547
22	.0364	.5612	.0370	.5677	.0375	.5741	.0381	.5805	.0386	.5868	.0392	.5931
23	.0397	.5993	.0403	.6055	.0409	.6116	.0415	.6177	.0421	.6238	.0426	.6298
24	.0432	.6357	.0438	.6417	.0444	.6476	.0450	.6534	.0456	.6592	.0462	.6650
25	.0468	.6707	.0475	.6764	.0481	.6820	.0487	.6876	.0493	.6932	.0500	.6987
26	.0506	.7042	.0512	.7096	.0519	.7151	.0525	.7204	.0532	.7258	.0538	.7311
27	.0545	.7364	.0552	.7416	.0558	.7468	.0565	.7520	.0572	.7572	.0578	.7623
28	.0585	.7673	.0592	.7724	.0599	.7774	.0606	.7824	.0613	.7874	.0620	.7923
29	.0627	.7972	.0634	.8020	.0641	.8069	.0648	.8117	.0655	.8165	.0663	.8213
30	.0670	.8260	.0677	.8307	.0684	.8354	.0692	.8400	.0699	.8446	.0707	.8492
31	.0714	.8538	.0722	.8583	.0729	.8629	.0737	.8673	.0744	.8718	.0752	.8763
32	.0760	.8807	.0767	.8851	.0775	.8894	.0783	.8938	.0791	.8981	.0799	.9024
33	.0807	.9067	.0815	.9109	.0823	.9152	.0831	.9194	.0839	.9236	.0847	.9277
34	.0855	.9319	.0863	.9360	.0871	.9401	.0879	.9442	.0888	.9482	.0896	.9523
35	.0904	.9563	.0913	.9603	.0921	.9643	.0929	.9682	.0938	.9722	.0946	.9761
36	.0955	.9800	.0963	.9838	.0972	.9877	.0981	.9915	.0989	.9954	.0998	.9992
37	.1007	.0030	.1016	.0067	.1024	.0105	.1033	.0142	.1042	.0179	.1051	.0216
38	.1060	.0253	.1069	.0289	.1078	.0326	.1087	.0362	.1096	.0398	.1105	.0434
39	.1114	.0470	.1123	.0505	.1133	.0541	.1142	.0576	.1151	.0611	.1160	.0646
40	.1170	.0681	.1179	.0716	.1189	.0750	.1198	.0784	.1207	.0817	.1217	.0853
41	.1226	.0887	.1236	.0920	.1246	.0954	.1255	.0987	.1265	.1021	.1275	.1054
42	.1284	.1087	.1294	.1119	.1304	.1152	.1314	.1185	.1323	.1217	.1333	.1249
43	.1343	.1282	.1353	.1314	.1363	.1345	.1373	.1377	.1383	.1409	.1393	.1440
44	.1403	.1472	.1413	.1503	.1424	.1534	.1434	.1565	.1444	.1596	.1454	.1626
45	.1464	.1657	.1475	.1687	.1485	.1718	.1495	.1748	.1506	.1778	.1516	.1808
46	.1527	.1838	.1538	.1867	.1548	.1897	.1558	.1926	.1569	.1956	.1579	.1985
47	.1590	.2014	.1600	.2043	.1611	.2072	.1622	.2101	.1633	.2129	.1644	.2158
48	.1654	.2186	.1665	.2215	.1676	.2243	.1687	.2271	.1698	.2299	.1709	.2327
49	.1720	.2355	1731	.2382	.1742	.2410	.1753	.2437	.1764	.2465	.1775	.2492
50	.1786	.2519	.1797	.2546	.1808	.2573	.1820	.2600	.1831	.2627	.1842	.2653
51	.1853	.2680	.1865	.2706	.1876	.2732	.1887	.2759	.1899	.2785	.1910	.2811
52	.1922	.2837	.1933	.2863	.1945	.2888	.1956	.2914	.1968	.2940	.1979	.2965
53	.1991	.2991	.2003	.3016	.2014	.3041	.2026	.3066	.2038	.3091	.2049	.3116
54	.2061	.3141	.2073	.3166	.2085	.3190	.2096	.3215	.2108	.3239	.2120	.3264
55	.2132	.3288	.2144	.3312	.2156	.3336	.2168	.3361	.2180	.3384	.2192	.3408
56	.2204	.3432	.2216	.3456	.2228	.3480	.2240	.3503	.2252	.3527	.2265	.3550
57	.2277	.3573	.2289	.3596	.2301	.3620	.2314	.3643	.2326	.3666	.2338	.3689
58	.2350	.3711	.2363	.3734	.2375	.3757	.2388	.3779	.2400	.3802	.2412	.3824
59	.2425	.3847	.2437	.3869	.2450	.3891	.2462	.3913	.2475	.3935	.2487	.3957

[Characteristics of Logarithms omitted — determine by rule from the value]

°	0′ Value	0′ Log_{10}	10′ Value	10′ Log_{10}	20′ Value	20′ Log_{10}	30′ Value	30′ Log_{10}	40′ Value	40′ Log_{10}	50′ Value	50′ Log_{10}
60	.2500	.3979	.2513	.4001	.2525	.4023	.2538	.4045	.2551	.4066	.2563	.4088
61	.2576	.4109	.2589	.4131	.2601	.4152	.2614	.4173	.2627	.4195	.2640	.4216
62	.2653	.4237	.2665	.4258	.2678	.4279	.2691	.4300	.2704	.4320	.2717	.4341
63	.2730	.4362	.2743	.4382	.2756	.4403	.2769	.4423	.2782	.4444	.2795	.4464
64	.2808	.4484	.2821	.4504	.2834	.4524	.2847	.4545	.2861	.4565	.2874	.4584
65	.2887	.4604	.2900	.4624	.2913	.4644	.2927	.4664	.2940	.4683	.2953	.4703
66	.2966	.4722	.2980	.4742	.2993	.4761	.3006	.4780	.3020	.4799	.3033	.4819
67	.3046	.4838	.3060	.4857	.3073	.4876	.3087	.4895	.3100	.4914	.3113	.4932
68	.3127	.4951	.3140	.4970	.3154	.4989	.3167	.5007	.3181	.5026	.3195	.5044
69	.3208	.5063	.3222	.5081	.3235	.5099	.3249	.5117	.3263	.5136	.3276	.5154
70	.3290	.5172	.3304	.5190	.3317	.5208	.3331	.5226	.3345	.5244	.3358	.5261
71	.3372	.5279	.3386	.5297	.3400	.5314	.3413	.5332	.3427	.5349	.3441	.5367
72	.3455	.5384	.3469	.5402	.3483	.5419	.3496	.5436	.3510	.5454	.3524	.5471
73	.3538	.5488	.3552	.5505	.3566	.5522	.3580	.5539	.3594	.5556	.3608	.5572
74	.3622	.5589	.3636	.5606	.3650	.5623	.3664	.5639	.3678	.5656	.3692	.5672
75	.3706	.5689	.3720	.5705	.3734	.5722	.3748	.5738	.3762	.5754	.3776	.5771
76	.3790	.5787	.3805	.5803	.3819	.5819	.3833	.5835	.3847	.5851	.3861	.5867
77	.3875	.5883	.3889	.5899	.3904	.5915	.3918	.5930	.3932	.5946	.3946	.5962
78	.3960	.5977	.3975	.5993	.3989	.6009	.4003	.6024	.4017	.6039	.4032	.6055
79	.4046	.6070	.4060	.6085	.4075	.6101	.4089	.6116	.4103	.6131	.4117	.6146
80	.4132	.6161	.4146	.6176	.4160	.6191	.4175	.6206	.4189	.6221	.4203	.6236
81	.4218	.6251	.4232	.6266	.4247	.6280	.4261	.6295	.4275	.6310	.4290	.6324
82	.4304	.6339	.4319	.6353	.4333	.6368	.4347	.6382	.4362	.6397	.4376	.6411
83	.4391	.6425	.4405	.6440	.4420	.6454	.4434	.6468	.4448	.6482	.4463	.6496
84	.4477	.6510	.4492	.6524	.4506	.6538	.4521	.6552	.4535	.6566	.4550	.6580
85	.4564	.6594	.4579	.6607	.4593	.6621	.4608	.6635	.4622	.6649	.4637	.6662
86	.4651	.6676	.4666	.6689	.4680	.6703	.4695	.6716	.4709	.6730	.4724	.6743
87	.4738	.6756	.4753	.6770	.4767	.6783	.4782	.6796	.4796	.6809	.4811	.6822
88	.4826	.6835	.4840	.6848	.4855	.6862	.4869	.6875	.4884	.6887	.4898	.6900
89	.4913	.6913	.4937	.6926	.4942	.6939	.4956	.6952	.4971	.6964	.4985	.6977
90	.5000	.6990	.5015	.7002	.5029	.7015	.5044	.7027	.5058	.7040	.5073	.7052
91	.5087	.7065	.5102	.7077	.5116	.7090	.5131	.7102	.5145	.7114	.5160	.7126
92	.5174	.7139	.5189	.7151	.5204	.7163	.5218	.7175	.5233	.7187	.5247	.7199
93	.5262	.7211	.5276	.7223	.5291	.7235	.5305	.7247	.5320	.7259	.5334	.7271
94	.5349	.7283	.5363	.7294	.5378	.7306	.5392	.7318	.5407	.7329	.5421	.7341
95	.5436	.7353	.5450	.7364	.5465	.7376	.5479	.7387	.5494	.7399	.5508	.7410
96	.5523	.7421	.5537	.7433	.5552	.7444	.5566	.7455	.5580	.7467	.5595	.7478
97	.5609	.7489	.5624	.7500	.5638	.7511	.5653	.7523	.5667	.7534	.5682	.7545
98	.5696	.7556	.5710	.7567	.5725	.7577	.5739	.7588	.5753	.7599	.5768	.7610
99	.5782	.7621	.5797	.7632	.5811	.7642	.5825	.7653	.5840	.7664	.5854	.7674
100	.5868	.7685	.5883	.7696	.5897	.7706	.5911	.7717	.5925	.7727	.5940	.7738
101	.5954	.7748	.5968	.7759	.5983	.7769	.5997	.7779	.6011	.7790	.6025	.7800
102	.6040	.7810	.6054	.7820	.6068	.7830	.6082	.7841	.6096	.7851	.6111	.7861
103	.6125	.7871	.6139	.7881	.6153	.7891	.6167	.7901	.6181	.7911	.6195	.7921
104	.6210	.7931	.6224	.7940	.6238	.7950	.6252	.7960	.6266	.7970	.6280	.7980
105	.6294	.7989	.6308	.7999	.6322	.8009	.6336	.8018	.6350	.8028	.6364	.8037
106	.6378	.8047	.6392	.8056	.6406	.8066	.6420	.8075	.6434	.8085	.6448	.8094
107	.6462	.8104	.6476	.8113	.6490	.8122	.6504	.8131	.6517	.8141	.6531	.8150
108	.6545	.8159	.6559	.8168	.6573	.8177	.6587	.8187	.6600	.8196	.6614	.8205
109	.6628	.8214	.6642	.8223	.6655	.8232	.6669	.8241	.6683	.8250	.6696	.8258
110	.6710	.8267	.6724	.8276	.6737	.8285	.6751	.8294	.6765	.8302	.6778	.8311
111	.6792	.8320	.6805	.8329	.6819	.8337	.6833	.8346	.6846	.8354	.6860	.8363
112	.6873	.8371	.6887	.8380	.6900	.8388	.6913	.8397	.6927	.8405	.6940	.8414
113	.6954	.8422	.6967	.8430	.6980	.8439	.6994	.8447	.7007	.8455	.7020	.8464
114	.7034	.8472	.7047	.8480	.7060	.8488	.7073	.8496	.7087	.8504	.7100	.8513
115	.7113	.8521	.7126	.8529	.7139	.8537	.7153	.8545	.7166	.8553	.7179	.8561
116	.7192	.8568	.7205	.8576	.7218	.8584	.7231	.8592	.7244	.8600	.7257	.8608
117	.7270	.8615	.7283	.8623	.7296	.8631	.7309	.8638	.7322	.8646	.7335	.8654
118	.7347	.8661	.7360	.8669	.7373	.8676	.7386	.8684	.7399	.8691	.7411	.8699
119	.7424	.8706	.7437	.8714	.7449	.8721	.7462	.8729	.7475	.8736	.7487	.8743

Values and Logarithms of Haversines

[Characteristics of Logarithms omitted — determine by rule from the value]

°	0 Value	0 Log₁₀	10′ Value	10′ Log₁₀	20′ Value	20′ Log₁₀	30′ Value	30′ Log₁₀	40′ Value	40′ Log₁₀	50′ Value	50′ Log₁₀
120	.7500	.8751	.7513	.8758	.7525	.8765	.7538	.8772	.7550	.8780	.7563	.8787
121	.7575	.8794	.7588	.8801	.7600	.8808	.7612	.8815	.7625	.8822	.7637	.8829
122	.7650	.8836	.7662	.8843	.7674	.8850	.7686	.8857	.7699	.8864	.7711	.8871
123	.7723	.8878	.7735	.8885	.7748	.8892	.7760	.8898	.7772	.8905	.7784	.8912
124	.7796	.8919	.7808	.8925	.7820	.8932	.7832	.8939	.7844	.8945	.7856	.8952
125	.7868	.8959	.7880	.8965	.7892	.8972	.7904	.8978	.7915	.8985	.7927	.8991
126	.7939	.8998	.7951	.9004	.7962	.9010	.7974	.9017	.7986	.9023	.7997	.9030
127	.8009	.9036	.8021	.9042	.8032	.9048	.8044	.9055	.8055	.9061	.8067	.9067
128	.8078	.9073	.8090	.9079	.8101	.9085	.8113	.9092	.8124	.9098	.8135	.9104
129	.8147	.9110	.8158	.9116	.8169	.9122	.8180	.9128	.8192	.9134	.8203	.9140
130	.8214	.9146	.8225	.9151	.8236	.9157	.8247	.9163	.8258	.9169	.8269	.9175
131	.8280	.9180	.8291	.9186	.8302	.9192	.8313	.9198	.8324	.9203	.8335	.9209
132	.8346	.9215	.8356	.9220	.8367	.9226	.8378	.9231	.8389	.9237	.8399	.9242
133	.8410	.9248	.8421	.9253	.8431	.9259	.8442	.9264	.8452	.9270	.8463	.9275
134	.8473	.9281	.8484	.9286	.8494	.9291	.8501	.9297	.8515	.9302	.8525	.9307
135	.8536	.9312	.8546	.9318	.8556	.9323	.8566	.9328	.8576	.9333	.8587	.9338
136	.8597	.9343	.8607	.9348	.8617	.9353	.8627	.9359	.8637	.9364	.8647	.9369
137	.8657	.9374	.8667	.9379	.8677	.9383	.8686	.9388	.8696	.9393	.8706	.9398
138	.8716	.9403	.8725	.9408	.8735	.9413	.8745	.9417	.8754	.9422	.8764	.9427
139	.8774	.9432	.8783	.9436	.8793	.9441	.8802	.9446	.8811	.9450	.8821	.9455
140	.8830	.9460	.8840	.9464	.8849	.9469	.8858	.9473	.8867	.9478	.8877	.9482
141	.8886	.9487	.8895	.9491	.8904	.9496	.8913	.9500	.8922	.9505	.8931	.9509
142	.8940	.9513	.8949	.9518	.8958	.9522	.8967	.9526	.8976	.9531	.8984	.9535
143	.8993	.9539	.9002	.9543	.9011	.9548	.9019	.9552	.9028	.9556	.9037	.9560
144	.9045	.9564	.9054	.9568	.9062	.9572	.9071	.9576	.9079	.9580	.9087	.9584
145	.9096	.9588	.9104	.9592	.9112	.9596	.9121	.9600	.9129	.9604	.9137	.9608
146	.9145	.9612	.9153	.9616	.9161	.9620	.9169	.9623	.9177	.9627	.9185	.9631
147	.9193	.9635	.9201	.9638	.9209	.9642	.9217	.9646	.9225	.9650	.9233	.9653
148	.9240	.9657	.9248	.9660	.9256	.9664	.9263	.9668	.9271	.9671	.9278	.9675
149	.9286	.9678	.9293	.9682	.9301	.9685	.9308	.9689	.9316	.9692	.9323	.9695
150	.9330	.9699	.9337	.9702	.9345	.9706	.9352	.9709	.9359	.9712	.9366	.9716
151	.9373	.9719	.9380	.9722	.9387	.9725	.9394	.9729	.9401	.9732	.9408	.9735
152	.9415	.9738	.9422	.9741	.9428	.9744	.9435	.9747	.9442	.9751	.9448	.9754
153	.9455	.9757	.9462	.9760	.9468	.9763	.9475	.9766	.9481	.9769	.9488	.9772
154	.9494	.9774	.9500	.9777	.9507	.9780	.9513	.9783	.9519	.9786	.9525	.9789
155	.9532	.9792	.9538	.9794	.9544	.9797	.9550	.9800	.9556	.9803	.9562	.9805
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157	.9603	.9824	.9608	.9826	.9614	.9829	.9619	.9831	.9625	.9834	.9630	.9836
158	.9636	.9839	.9641	.9841	.9647	.9844	.9652	.9846	.9657	.9849	.9663	.9851
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165	.9830	.9925	.9833	.9927	.9837	.9929	.9841	.9930	.9844	.9932	.9848	.9933
166	.9851	.9935	.9855	.9937	.9858	.9938	.9862	.9940	.9865	.9941	.9869	.9943
167	.9872	.9944	.9875	.9945	.9878	.9947	.9881	.9948	.9885	.9950	.9888	.9951
168	.9891	.9952	.9894	.9954	.9897	.9955	.9900	.9956	.9903	.9957	.9905	.9959
169	.9908	.9960	.9911	.9961	.9914	.9962	.9916	.9963	.9919	.9965	.9921	.9966
170	.9924	.9967	.9927	.9968	.9929	.9969	.9931	.9970	.9934	.9971	.9936	.9972
171	.9938	.9973	.9941	.9974	.9943	.9975	.9945	.9976	.9947	.9977	.9949	.9978
172	.9951	.9979	.9953	.9980	.9955	.9981	.9957	.9981	.9959	.9982	.9961	.9983
173	.9963	.9984	.9964	.9984	.9966	.9985	.9968	.9986	.9969	.9987	.9971	.9987
174	.9973	.9988	.9974	.9988	.9976	.9989	.9977	.9990	.9978	.9991	.9980	.9991
175	.9981	.9992	.9982	.9992	.9983	.9993	.9985	.9993	.9986	.9994	.9987	.9994
176	.9988	.9995	.9989	.9995	.9990	.9996	.9991	.9996	.9992	.9996	.9992	.9997
177	.9993	.9997	.9994	.9997	.9995	.9998	.9995	.9998	.9996	.9998	.9996	.9998
178	.9997	.9999	.9997	.9999	.9998	.9999	.9998	.9999	.9999	.9999	.9999	.9999
179	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0.0000	1.0000	.0000

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$S = P(1+i)^n$

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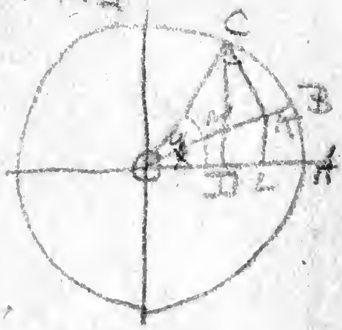
$\sin x+y = \sin x \cos y + \cos x \sin y$.



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(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $\tan^2\theta + 1 = \sec^2\theta$

(3) $\cot^2\theta + 1 = \csc^2\theta$

(4) $\frac{\sin\theta}{\cos\theta} = \tan\theta$

(5) $\frac{\cos\theta}{\sin\theta} = \cot\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

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